

CHARACTERIZATION OF $\mathbb{A}^1$ -MILNOR NUMBERS OF SMALL RANK

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ABSTRACT. The $\mathbb{A}^1$ -Milnor number of an isolated algebraic singularity is a quadratic enrichment of the classical Milnor number. In this paper, we study which symmetric bilinear forms of small rank arise as $\mathbb{A}^1$ -Milnor numbers. We show that every Grothendieck–Witt class of rank at most 7 that arises as an EKL form also arises as an $\mathbb{A}^1$ -Milnor number. In general, however, not every EKL form arises as an $\mathbb{A}^1$ -Milnor number of an isolated singularity of a plane algebraic curve. We characterize the Grothendieck–Witt classes of rank at most 8 that arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities of plane algebraic curves.

1. INTRODUCTION

The classical Milnor number [14] is an important invariant of an isolated singularity of a complex algebraic hypersurface. The celebrated $\mathbb{A}^1$ -homotopy theory of Morel and Voevodsky [15, 16] has allowed the generalization of many topological invariants. Specifically, the classical Brouwer degrees and Milnor numbers over $\mathbb{C}$ have been generalized to maps over an arbitrary field k . In this generalization, these invariants are no longer numbers, but are now classes in the Grothendieck–Witt group $GW(k)$ of symmetric bilinear forms over k .

Let $F: \mathbb{A}^n \rightarrow \mathbb{A}^n$ be a morphism with an isolated zero at the origin. Recent work of Kass–Wickelgren [9] shows that $\deg_0(F)$, the local $\mathbb{A}^1$ -Brouwer degree of F , is equal to an *EKL form*, which is a Grothendieck–Witt class that was previously defined by Eisenbud–Levine [5] and Khimshiashvili [7].

If a hypersurface $V(f)$ is defined by $f = 0$ for some morphism $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$, then the gradient of f gives a map $\nabla f: \mathbb{A}^n \rightarrow \mathbb{A}^n$. Assume that 0 is an isolated singular point of $V(f)$. Then the $\mathbb{A}^1$ -Milnor number (over an arbitrary field) of $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$ is defined by

$$\mu_0^{\mathbb{A}^1}(f) = \deg_0^{\mathbb{A}^1}(\nabla f)$$

the local $\mathbb{A}^1$ -Brouwer degree of ∇f [9, p. 3]. This definition is consistent with the classical case (over $\mathbb{C}$), when the Milnor number of f at an isolated singularity agrees with the local Brouwer degree of ∇f at that point. By Kass and Wickelgren [9], $\mu_0^{\mathbb{A}^1}(f)$ is an EKL form. This paper studies which EKL forms and GW-classes arise as $\mathbb{A}^1$ -Milnor numbers of isolated algebraic singularities.

Quick–Strand–Wilson [17] have classified which Grothendieck–Witt classes of rank at most 7 are realized as EKL forms. In particular, not every Grothendieck–Witt class arises as an EKL form. It is natural to ask whether every EKL form arises as an $\mathbb{A}^1$ Milnor number of some such f . In other words, does every EKL form arise as the EKL form of ∇f for some $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$? We answer this question in the affirmative for rank at most 7 by explicitly constructing trivariate polynomials that define hypersurface singularities that have as $\mathbb{A}^1$ -Milnor numbers all of the EKL forms found by [17].

Theorem 3.3. *Let k be a field of characteristic not 2, 3, 5, or 7 and let $q \in \mathrm{GW}(k)$ be a class of rank ≤ 7 . If q arises as an EKL form, then q is an $\mathbb{A}^1$ -Milnor number for some $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$ with $n \leq 3$.*

In the work of Quick–Strand–Wilson, all the Grothendieck–Witt classes of rank at most 7 arise as EKL forms from morphisms $\mathbb{A}^2 \rightarrow \mathbb{A}^2$. It is natural to ask the analogous question for $\mathbb{A}^1$ -Milnor numbers and whether Theorem 3.3 can be strengthened to require $n \leq 2$. However, this cannot occur. In Theorems 8.1, we show that there are $\mathbb{A}^1$ -Milnor numbers of ranks 4, 6, and 8 over $\mathbb{Q}$ and $\mathbb{R}$ that do not arise from bivariate polynomials $f: \mathbb{A}^2 \rightarrow \mathbb{A}^1$.

The bulk of this paper is focused on finding all Grothendieck–Witt classes of rank at most 8 that arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities of plane algebraic curves. In the case of ranks 1, 2, 3, 5, and 7, the examples of Section 3 show that all Grothendieck–Witt classes that arise as EKL forms also arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities of plane algebraic curves. In subsequent sections, we thus restrict attention to ranks 4, 6, and 8.

In Section 4, we give necessary background about Newton nondegeneracy. In Section 5, we present some theorems concluded from studying $\mathbb{A}^1$ -Milnor numbers in general. A result of Quick–Strand–Wilson [17, Lemma 12] allows us to restrict attention to polynomials $f(x, y)$ that include terms x^m and y^n for sufficiently large m and n . Considering the Newton diagram of such a polynomial, its Newton number gives a lower bound on the rank of its $\mathbb{A}^1$ -Milnor number. Moreover, Newton nondegeneracy implies equality. In Theorem 5.3, we prove that $\mathbb{A}^1$ -Milnor numbers are invariant under a change of variables corresponding to . As a consequence, in Theorem 5.7, we show that, under suitable hypotheses, if a polynomial is not Newton nondegenerate, then it has the same $\mathbb{A}^1$ -Milnor number as a polynomial with larger Newton number. This result allows us to restrict our attention to polynomials that are Newton nondegenerate.

In Section 6, we turn our attention to cataloging the Newton diagrams that have Newton number at most 8. It turns out that the hypotheses of Theorem 5.7 are always satisfied for rank at most 8, thus allowing us to restrict attention to Newton nondegenerate polynomials. These Newton diagrams determine the possible forms of polynomials that have $\mathbb{A}^1$ -Milnor number of rank at most 8. In Section 7, we directly compute the $\mathbb{A}^1$ -Milnor number of all polynomials with such Newton diagrams. As a consequence, we can determine all Grothendieck–Witt classes of rank at most 8 that arise as an $\mathbb{A}^1$ -Milnor number of a plane algebraic curves, as shown in Table 1.

In Section 8, we consider the special cases when $k = \mathbb{Q}$ or $\mathbb{R}$. We prove that there is no $\mathbb{A}^1$ -Milnor number of an isolated singularity of a plane algebraic curve that has signature $(3, 1)$, $(4, 2)$, or $(5, 3)$. As there are EKL forms with each of these signatures, this shows that not all EKL forms arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities of plane algebraic curves (Theorem 8.1).

In Section 9, we prove an analogue for $\mathbb{A}^1$ -Milnor numbers of Quick–Strand–Wilson’s reduction theorem [17, Theorem 15] for EKL forms, a result that may be of independent interest to some readers. This result follows from an application of Theorem 5.3 on reducing the number of variables in the function under consideration and preserving the $\mathbb{A}^1$ -Milnor number up to multiplication by a unit. Theorem 9.6 states that under some minor condition, there exists g of few variables than f such that $\mu_0^{\mathbb{A}^1}(f)$ and $\mu_0^{\mathbb{A}^1}(g)$ are equivalent classes up to a unit multiplication. Such techniques can also be seen in McKean’s calculation [13, Lemma 5.7] of the local $\mathbb{A}^1$ -Brouwer degrees of the intersection of two plane curves.

Rank	$\mathbb{A}^1$ -Milnor numbers
1	$\langle a \rangle$
2	h
3	$h + \langle a \rangle$
4	$h + \langle 3\Delta_g A, A \rangle$
5	$2h + \langle a \rangle$
6	$2h + \langle -2, c \rangle$
7	$3h + \langle a \rangle$
8	$3h + \langle -2, c \rangle$

TABLE 1. Possible $\mathbb{A}^1$ -Milnor numbers $\mu_0^{\mathbb{A}^1}(f)$ for bivariate f . Here a and c are allowed to take any nonzero value, and Δ_g and A are as defined in Section 7.2.

2. PRELIMINARIES

In this section, we review some facts about $\mathbb{A}^1$ -Milnor numbers and $\mathbb{A}^1$ -Brouwer degrees. We follow the approach of Kass and Wickelgren [9], which computes EKL forms over a localized algebra. This differs from the alternative approach taken in [3], which instead uses saturation and computes Bézoutian bilinear forms.

Let k be a field and let $\mathbb{A}_k^n$ denote affine n -space over k . For an algebraic set $X \subseteq \mathbb{A}^n$, a map $f: X \rightarrow \mathbb{A}^1$ is a *morphism* (or equivalently a regular map) if f is a restriction of a polynomial function in n variables (i.e. f is a regular function on X). When $X = \mathbb{A}^n$, one can assume that f is given by a polynomial function in n variables over k (cf. Eisenbud [4, Section 1.6, p. 36]). Let $f \in k[x_1, \dots, x_n]$. The *hypersurface* $V(f)$ is the solution set of $f = 0$ in $\mathbb{A}^n$. A closed point p on $V(f)$ is a *singular* point if $f_{x_i}(p) = 0$ for all $i \in [n]$. The set $\text{Sing}(f)$ is the set of singular points of $V(f)$, also referred to as the *singularities* of $V(f)$, and the elements of $V(f) \setminus \text{Sing}(f)$ are the regular points of $V(f)$. An *isolated singularity* of $V(f)$ is a closed point $p \in \text{Sing}(f)$ such that there is a neighborhood of p in $V(f)$ in which p is the only singularity. Algebraically, this means that the localization $k[x_1, \dots, x_n]_p / (f, \nabla f)$ is a finite dimensional k -vector space. The main case in this paper is when $p = 0 \in \text{Sing}(f)$, and we write that f has an isolated singularity at the origin.

Let $F: \mathbb{A}^n \rightarrow \mathbb{A}^n$ be a morphism with an zero at the origin. Let $\mathfrak{m}$ be the maximal ideal $(x_1, \dots, x_n)$, and let $\mathcal{I}_F = (F_1, \dots, F_n)$ be the ideal of $k[x_1, \dots, x_n]$ generated by the coordinate functions F_i of F . The algebra $Q_0(F)$ is defined as $(k[x_1, \dots, x_n] / \mathcal{I}_F)_{\mathfrak{m}}$, the localization of the quotient ring at $\mathfrak{m}$. We say that F has an isolated zero at the origin if $Q_0(F)$ is a finite-dimensional k -vector space.

In the case of a morphism $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$ with an isolated singularity at 0, we apply the above construction to $\nabla f: \mathbb{A}^n \rightarrow \mathbb{A}^n$. If $Q_0(\nabla f)$ is a finite-dimensional vector space, the Milnor number $\mu_0(f)$ is defined to be $\dim_k Q_0(\nabla f)$. Otherwise, we say that $\mu_0(f) = \infty$. When $k = \mathbb{C}$, Milnor [14] proved that $\mu_0(f)$ is finite by a topological argument.

The Milnor number $\mu_0(f)$ can be defined over an arbitrary field k , but it is not necessarily finite. When k has characteristic zero, Milnor [14, Problem 3, page 115] indicated an algebraic proof for the finiteness of $\mu_0(f)$. ~~Another algebraic proof in the characteristic zero setting follows from [8, Corollary 7.1.4, Theorem 7.1.5] (See also [6, Sections 2 and 3] for a detailed discussion).~~

However, if k has positive characteristic, the Milnor number is not necessarily finite. For instance, $f(x, y) = x^2y + xy^2$ over a field k of characteristic 3 (see Hefez et.al. [6]) has an isolated singularity at the origin but infinite Milnor number. However, if f satisfies the additional condition of non-degeneracy (see Section 4), then $\mu_0(f)$ is finite [11].

The *Grothendieck–Witt ring* of a field is the group completion of the set of isometry classes of nondegenerate symmetric bilinear forms, with addition given by direct sum and multiplication given by the tensor product.

Given two symmetric bilinear forms $\beta_i: V_i \times V_i \rightarrow k$ for $i = 1, 2$, we can define their sum and product:

$$(1) \quad \begin{aligned} (\beta_1 \oplus \beta_2): (V_1 \oplus V_2) \times (V_1 \oplus V_2) &\rightarrow k \\ (\beta_1 \otimes \beta_2): (V_1 \otimes V_2) \times (V_1 \otimes V_2) &\rightarrow k. \end{aligned}$$

On Gram matrices, these operations are given by direct sum and tensor product, respectively.

If $\{x_1, \dots, x_n\}$ is a basis of Q then the *Gram matrix* of a bilinear form $\langle \cdot, \cdot \rangle$ on Q is defined to be the matrix whose i, j entry is given by $\langle x_i, x_j \rangle$. The *hyperbolic form* is the symmetric bilinear form whose Gram matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; we denote the hyperbolic form by h . For more details, see [12].

In $\mathbb{A}^1$ -homotopy theory, there are quadratic enrichments of Milnor numbers and Brouwer degrees. Morel [15, Corollary 1.24] constructs a canonical morphism

$$[\mathbb{A}^n - \{0\}, \mathbb{A}^n - \{0\}]_{\bullet} \rightarrow \mathrm{GW}(k),$$

which defines the local $\mathbb{A}^1$ -Brouwer degree $\deg_0^{\mathbb{A}^1}$. The $\mathbb{A}^1$ -Milnor number is then defined in terms of the local $\mathbb{A}^1$ -Brouwer degree as follows.

Definition 2.1. Suppose the hypersurface defined by $f = 0$ for $f \in k[x_1, \dots, x_n]$ has an isolated singularity at 0. The $\mathbb{A}^1$ -Milnor number $\mu_0^{\mathbb{A}^1}(f)$ of f at 0 is defined as

$$\mu_0^{\mathbb{A}^1}(f) = \deg_0^{\mathbb{A}^1}(\nabla f).$$

The $\mathbb{A}^1$ -Brouwer degree is a Grothendieck–Witt class given by the so-called EKL form, as we will now explain. Eisenbud–Levine [5] and Khimshiashvili [7] independently defined a symmetric bilinear form now known as an EKL form. It was shown by Kass–Wickelgren [9] that the local $\mathbb{A}^1$ -Brouwer degree at an isolated singularity of a morphism $\mathbb{A}^n \rightarrow \mathbb{A}^n$ coincides with the corresponding EKL form of the morphism. Next we explain how to compute the appropriate EKL form to find the *local $\mathbb{A}^1$ -Brouwer degree*.

which showed that $\mu_0(f)$ equals the local Brouwer degree of a hypersurface at the origin.

Given a map $F: \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^n$ with an isolated zero at 0, define a map $S^{2n-1} \rightarrow S^{2n-1}$ by taking a small ball around 0 and mapping it by $x \mapsto \frac{F(x)}{\|F(x)\|}$. The local Brouwer degree of F at x is then defined to be the ordinary Brouwer degree of this map.

Such a form, denoted $w(F)$, was originally constructed as a symmetric bilinear form for a C^∞ -function F with an isolated zero [5, 7]. The classical local Brouwer degree of F is recovered from the signature of $w(F)$ when F is a C^∞ real-valued function, and from the rank of $w(F_{\mathbb{C}})$ of the complexification $F_{\mathbb{C}}$ when F is a real analytic function. The definition of $w(F)$ remains valid when F is a polynomial function with coefficient in an arbitrary field. Eisenbud asked if the Grothendieck–Witt class $w(F)$ can be identified with some degree in algebraic topology, which Kass and Wickelgren [9] then answered by proving that $w(F)$ is the local $\mathbb{A}^1$ -Brouwer degree. This relationship provides a tractable way to compute local $\mathbb{A}^1$ -Brouwer degrees and hence $\mathbb{A}^1$ -Milnor numbers.

Describe GW ring/classes with more helpful/precise details: such as equivalence, addition, multiplication, units...

Let Q be a k -algebra that is a commutative ring of Krull dimension zero. In this case, Q can be viewed as a finite-dimensional vector space over k . The *socle* of Q is the sum of all simple submodules of Q . Then Q is Gorenstein if and only if the socle of Q is a principal ideal. The generator of the socle of a Gorenstein module, up to a unit, is called the *distinguished socle element*.

In [5], a nonsingular linear functional $\varphi: Q \rightarrow k$ defines a bilinear form by setting

$$\langle p, q \rangle = \langle p, q \rangle_\varphi := \varphi(pq)$$

for any $p, q \in Q$. By [5, Proposition 3.1], such a functional is nonsingular if and only if it is a generator of $Q^* := \text{Hom}_k(Q, k)$. The same proposition also states that the existence of such a nonsingular functional is equivalent to $Q^* = Q$, and this is equivalent to Q being Gorenstein. Furthermore, [5, Proposition 3.5] proves that if ψ is also a nonsingular functional such that $\varphi(a)\psi(a)$ is a square in k for a fixed element a in the socle of Q , then $\langle \cdot, \cdot \rangle_\varphi = \langle \cdot, \cdot \rangle_\psi$. In other words, this bilinear form is, to certain extent, independent of the choice of linear functional.

For $Q = (k[x_1, \dots, x_n]/F)_{\mathfrak{m}}$, localized at the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$, one has that $E = \det(F_{i,j})$, where the $F_{i,j}$ are polynomials for which $F_i = \sum_{j=1}^n F_{i,j}x_j$ in Q . Up to a unit, E is independent of the choices of $F_{i,j}$. Kass and Wickelgren [9, Lemma 4] show, with reference to [18, (4.7) Korollar], that the socle of Q is generated by E . Therefore Q is Gorenstein, and if two linear functionals take the same value on E (or compatible values stated in [5, Proposition 3.5] also as just described above), they define equivalent bilinear forms. These equivalence relations provides a well-defined class in Grothendieck–Witt ring of symmetric bilinear forms. The EKL form of F corresponds to a k -linear functional $\varphi: Q \rightarrow k$ for which $\varphi(E) = 1$.

Let J_F denote the Jacobian matrix of F . It is also shown in [9] that E is equal to the $r \cdot \det J_F$ in Q , where r is the dimension of Q as a k -vector space. Hence, in the computations in the following sections, we define a k -functional η by setting $\eta(\det J_F) = r = \dim_k Q$. Note that

$$\varphi(E)\eta(E) = \varphi(E)\eta(r \cdot \det J_F) = 1 \cdot r \cdot \eta(\det J_F) = r^2.$$

This shows that φ and η defines the same bilinear function. In particular, we can obtain the EKL form by constructing the Grothendieck–Witt class using η . When finding an $\mathbb{A}^1$ -Milnor number, $J_{\nabla f}$ is the same as the *Hessian matrix* of f .

By fixing a basis for $Q_0(F)$, $\langle \cdot, \cdot \rangle_\psi$ can be represented by a symmetric $r \times r$ matrix. The equivalence relation resulting from the valid functionals allows us to view this bilinear form as an equivalence class in the Grothendieck–Witt ring of bilinear forms which is called the *EKL form* of F .

We now recall some basic results on symmetric bilinear forms that we will use in computations throughout the paper. For a symmetric matrix B , we let q_B denote the corresponding bilinear form.

Lemma 2.2. *Let q be a symmetric bilinear form with the upper antitriangular block matrix*

$$\left[\begin{array}{c|ccc|c} c_{11} & c_{12} & \cdots & c_{1,n-1} & a \\ c_{12} & & & & 0 \\ \vdots & & B & & \vdots \\ c_{1,n-1} & & & & 0 \\ \hline a & 0 & \cdots & 0 & 0 \end{array} \right]$$

with $a \neq 0$ and B is $(n-2) \times (n-2)$ square matrix. Then $q \cong h + q_B$.

While Lemma 2.2 is similar to [2, Lemma 4.5] and [1, Proposition 5.2], it isn't covered by their hypothesis and we present a proof.

Proof. For all j satisfying $2 \leq j \leq n-1$, subtract c_{1j}/a times the n th row from the j th row and c_{1j}/a times the n th column from the j th column. Then subtract $c_{11}/2a$ times the n th row from the first row and $c_{11}/2a$ times the n th column from the first column. This gives the matrix

$$\left[\begin{array}{c|ccc|c} 0 & 0 & \cdots & 0 & a \\ \hline 0 & & & & 0 \\ \vdots & & B & & \vdots \\ 0 & & & & 0 \\ \hline a & 0 & \cdots & 0 & 0 \end{array} \right].$$

Reordering rows and columns, this becomes

$$\left[\begin{array}{cc|ccc} 0 & a & 0 & \cdots & 0 \\ a & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & B & \\ 0 & 0 & & & \end{array} \right].$$

□

Repeatedly apply Lemma 2.2, we obtain the corollary proved in [10, Lemma 6].

Corollary 2.3. *Let B be the upper anti-triangular matrix*

$$\begin{bmatrix} * & \cdots & * & a \\ \vdots & \ddots & \ddots & 0 \\ * & \ddots & \ddots & \vdots \\ a & 0 & \cdots & 0 \end{bmatrix}$$

with a constant term $a \neq 0$ along the antidiagonal. Then

$$q_B \cong \begin{cases} \frac{m}{2}h & \text{if } m \text{ is even;} \\ \frac{m-1}{2}h + \langle a \rangle & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Repeatedly apply Lemma 2.2. □

We now provide an explicit example that illustrates many aspects of this paper.

Example 2.4. Let $f = (x + y^2)^2 + xy^3$. The Jacobian ideal of f is given by

$$J_f = (f_x, f_y) = (2x + 2y^2 + y^3, 4xy + 3xy^2 + 4y^3).$$

As f_x and f_y have no common zero, f has an isolated singularity at the origin. Hence $\mu_0(f) < \infty$.

We now find a basis for $Q_0(\nabla f) = (k[x, y]/J_f)_{(x, y)}$. Using $f_x = 0$, we substitute for x in $f_y = 0$ to obtain

$$J_f = (2x + 2y^2 + y^3, -y^4(10 + 3y)/2) = (2x + 2y^2 + y^3, y^4)$$

as an ideal in $k[x, y]_{(x, y)}$, as $10 + 3y$ is a unit. Then, in $Q_0(\nabla f)$, x can be written in terms of y , and $4y^4 = 0$. Thus $\mathcal{B} = \{1, y, y^2, y^3\}$ is a spanning set for $Q_0(\nabla f)$, and it can be shown to be a basis. Thus $\mu_0(f) = \dim Q_0(\nabla f) = 4$.

With the basis $\mathcal{B}$, we now calculate the $\mathbb{A}^1$ -Milnor number. The determinant of the Hessian matrix H_f of f is

$$\det H_f = 8x + 12xy + 8y^2 - 24y^3 - 9y^4 = -40y^3$$

in $Q_0(\nabla f)$ as $y^4 = 0$ and $x = -(2y^2 + y^3)/2$. Since $\mu_0(f) = 4$, we define $\eta(y^3) = -1/10$ and let $\eta = 0$ on the other basis elements of $\mathcal{B}$. Then $\eta(-40y^3) = 4$, as needed for the construction of $\mu_0^{\mathbb{A}^1}(f)$. The Gram matrix of $\mu_0^{\mathbb{A}^1}(f)$ is then

$$\begin{bmatrix} \eta(1 \cdot 1) & \eta(1 \cdot y) & \eta(1 \cdot y^2) & \eta(1 \cdot y^3) \\ \eta(y \cdot 1) & \eta(y \cdot y) & \eta(y \cdot y^2) & \eta(y \cdot y^3) \\ \eta(y^2 \cdot 1) & \eta(y^2 \cdot y) & \eta(y^2 \cdot y^2) & \eta(y^2 \cdot y^3) \\ \eta(y^3 \cdot 1) & \eta(y^3 \cdot y) & \eta(y^3 \cdot y^2) & \eta(y^3 \cdot y^3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{10} \\ 0 & 0 & -\frac{1}{10} & 0 \\ 0 & -\frac{1}{10} & 0 & 0 \\ -\frac{1}{10} & 0 & 0 & 0 \end{bmatrix}.$$

By Corollary 2.3, we have $\mu_0^{\mathbb{A}^1}(f) = 2h$.

We conclude this section with the following lemma, which shows how the $\mathbb{A}^1$ -Milnor number of a singularity is affected by multiplying its defining polynomial by a constant.

Lemma 2.5. *If $f: \mathbb{A}^n \rightarrow \mathbb{A}$ is a polynomial with isolated singularity of finite Milnor number at the origin. If a is a nonzero constant, then*

$$\mu_0^{\mathbb{A}^1}(af) = \begin{cases} \mu_0^{\mathbb{A}^1}(f) & \text{if } n \text{ is even} \\ \langle a \rangle \cdot \mu_0^{\mathbb{A}^1}(f) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $g = af$. Then

$$J_g = (g_{x_1}, \dots, g_{x_n}) = (af_{x_1}, \dots, af_{x_n}) = (f_{x_1}, \dots, f_{x_n}) = J_f,$$

and $Q_0(\nabla g) = Q_0(\nabla f)$. Since $H_g = aH_f$, we have $\det H_g = a^n \det H_f$. Defining $\eta_g = \frac{1}{a^n} \eta_f$, we have

$$\eta_g(\det H_g) = \frac{1}{a^n} \eta_f(a^n \det H_f) = \eta_f(\det H_f) = \dim_k Q_0(\nabla f) = \dim_k Q_0(\nabla g).$$

Thus the Gram matrix for g is $\frac{1}{a^n}$ times the one for f , and

$$\mu_0^{\mathbb{A}^1}(g) = \left\langle \frac{1}{a^n} \right\rangle \cdot \mu_0^{\mathbb{A}^1}(f) = \langle a^n \rangle \cdot \mu_0^{\mathbb{A}^1}(f).$$

Since $\langle a^n \rangle = \langle 1 \rangle$ when n is even and $\langle a^n \rangle = \langle a \rangle$ when n is odd, the result follows. $\square$

3. ALL EKL FORMS OF RANK AT MOST 7 ARISE AS $\mathbb{A}^1$ -MILNOR NUMBERS

Quick–Strand–Wilson [17] determine the Grothendieck–Witt classes of rank at most 7 which arise as EKL forms. These classes are shown in Table 2. In this section, we show that all of these forms also arise as $\mathbb{A}^1$ -Milnor numbers. We do this by explicitly calculating $\mu_0^{\mathbb{A}^1}(f)$ for families of polynomials f with isolated singularities at zero.

We first use a construction from [17] to show that many EKL forms arise as $\mathbb{A}^1$ -Milnor numbers for morphisms $f: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ with an isolated singularity at zero.

Example 3.1 (cf. [17, Lemma 5]). Let $f(x) = \frac{1}{a(m+1)}x^{m+1}$. We have $f_x = \frac{1}{a}x^m$ and $k[x]/(f_x)$ is an m -dimensional vector space with basis $\{1, x, \dots, x^{m-1}\}$. The Hessian determinant is $(\frac{m}{a})x^{m-1}$, and we define η by $\eta(x^{m-1}) = a$ and $\eta(x^i) = 0$ for all $i \in$

Rank	EKL forms
1	$\langle a \rangle$
2	h
3	$h + \langle a \rangle$
4	$h + \langle a, b \rangle$
5	$2h + \langle a \rangle$
6	$2h + \langle a, b \rangle$
7	$3h + \langle a \rangle$

TABLE 2. Possible EKL forms according to [17]. Here a and b are allowed to take any nonzero values.

$\{0, 1, \dots, m-2\}$. The Gram matrix is an antidiagonal matrix with a 's along the antidiagonal, so, by Corollary 2.3,

$$\mu_0^{\mathbb{A}^1}(f) = \begin{cases} \left(\frac{m}{2}\right) h & \text{if } m \text{ is even,} \\ \left(\frac{m-1}{2}\right) h + \langle a \rangle & \text{if } m \text{ is odd.} \end{cases}$$

The next example constructs families of singularities for surfaces defined by $f = 0$ for trivariate polynomials f .

Example 3.2. Assume $a, b \neq 0$ and let m be an even integer ≥ 4 . Consider the polynomial

$$f(x, y, z) = x^2y - \frac{a}{b(m-1)}y^{m-1} - \frac{1}{4a}z^2,$$

The Jacobian

$$J_f = (f_x, f_y, f_z) = \left(2xy, x^2 - \frac{a}{b}y^{m-2}, -\frac{1}{2a}z\right).$$

shows that f has an isolated singularity at the origin. Let $Q_0 = Q_0(\nabla f)$. J_f shows that $z = 0$ and $xy = 0$ in Q_0 . Then $xy^k = 0$ for $k \geq 1$. Since $xf_y = 0$ in Q_0 and $xy^{m-2} = 0$ as $m \geq 4$, one has $x^3 = 0$ in Q_0 . Similarly, $yf_y = 0$ and $x^2y = 0$ shows $y^k = 0$ for $k \geq m-1$. Also, $f_y \in J_f$ gives $x^2 = \frac{a}{b}y^{m-2}$. Hence, Q_0 is spanned by the set of m elements: $\{1, x, y, y^2, \dots, y^{m-2}\}$, which gives a m -dimensional basis for Q_0 . We will use the ordered basis

$$\mathcal{B} = \{1, \dots, y^{m/2-1}, x, y^{m/2}, \dots, y^{m-2}\}$$

The Hessian determinant $\det H_f$ is given by

$$\begin{vmatrix} 2y & 2x & 0 \\ 2x & -\frac{(m-2)a}{b}y^{m-3} & 0 \\ 0 & 0 & -\frac{1}{2a} \end{vmatrix} = \frac{m}{b}y^{m-2}.$$

Therefore we define η by $\eta(y^{m-2}) = b$ and $\eta = 0$ on the other elements of $\mathcal{B}$. The Gram matrix G is then the antidiagonal block matrix, with b 's and the matrix B on the antidiagonal,

with

$$G = \begin{bmatrix} & & & & b \\ & 0 & & & \\ & & & \ddots & \\ & & b & & \\ & & & B & \\ b & & & & 0 \end{bmatrix}, \quad \text{with } B = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$$

By applying Lemma 2.2 $(m-2)/2$ times, we have $\mu_0^{\mathbb{A}^1}(f) = \binom{m-2}{2} h + \langle a, b \rangle$.

Taken together, these examples show that every EKL form of rank at most 7 arises as an $\mathbb{A}^1$ -Milnor number.

Theorem 3.3. *Let k be a field of characteristic not 2, 3, 5, or 7 and let $q \in \text{GW}(k)$ be a class of rank ≤ 7 . If q arises as an EKL form, then q is an $\mathbb{A}^1$ -Milnor number for some $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$ with $n \leq 3$.*

Proof. Table 2 gives the Grothendieck–Witt classes of rank at most 7 that arise as EKL forms. Example 3.1 shows that every EKL form of rank 1, 2, 3, 5, and 7 arises as a $\mathbb{A}^1$ -Milnor number. Example 3.2 with $m = 4$ shows that every EKL form of rank 4 arises as an $\mathbb{A}^1$ -Milnor number. Example 3.2 with $m = 6$ shows that every EKL form of rank 6 arises as an $\mathbb{A}^1$ -Milnor number. $\square$

Theorem 3.3 shows that, up to rank 7, all Grothendieck–Witt classes that arise as EKL forms also arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities defined by polynomials in three or fewer variables. The construction in the following example can be used to show that once a Grothendieck–Witt class arises from a polynomial with a certain number of variables, then it also arises from polynomials with any larger number of variables.

Example 3.4. Given a polynomial $g(\mathbf{x})$, define $f(\mathbf{x}, z) = g(\mathbf{x}) + cz^2$, where c is a nonzero constant. The $\mathbb{A}^1$ -Milnor numbers of f and g are related by

$$\mu_0^{\mathbb{A}^1}(g) = \langle 2c \rangle \cdot \mu_0^{\mathbb{A}^1}(f),$$

We will generalize this in Theorem 9.6.

Theorem 3.5. (a) *Let m be a positive integer. If a Grothendieck–Witt class arises as the $\mathbb{A}^1$ -Milnor number of an isolated singularity of a polynomial in m variables, then, for all $n \geq m$, it also arises as the $\mathbb{A}^1$ -Milnor number of a polynomial in n variables.*
 (b) *Each EKL form of rank 1, 2, 3, 5, or 7 arises from a morphism $f: \mathbb{A}^1 \rightarrow \mathbb{A}^1$.*
 (c) *Each EKL form of rank 1, 2, 3, 5, or 7 arises from a morphism $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$, for every $n \geq 1$.*

Proof. Repeatedly applying the construction in Example 3.4 with $c = 1/2$ shows (a).

Table 2 gives the Grothendieck–Witt classes of ranks 1, 2, 3, 5, and 7 that arise as EKL forms. Example 3.1 shows that each of these forms arises as the $\mathbb{A}^1$ -Milnor number of a univariate polynomial, showing (b).

Part (c) follows from parts (a) and (b). $\square$

For the remainder of the paper, we study which Grothendieck–Witt classes arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities defined by single or two-variable (bivariate) polynomials. The following proposition generalizes Example 3.1 to determine the $\mathbb{A}^1$ -Milnor number when f is a single-variable function with an isolated singularity at zero.

Proposition 3.6. *Let $f = \sum_{i=d}^m a_i x^i \in k[x]$ with $a_d \neq 0$ be a polynomial such that $f: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ has an isolated singularity at zero. Then $d \geq 2$, $\mu_0(f) = d - 1$, and*

- (a) *If $d - 1$ is even, then the $\mathbb{A}^1$ -Milnor number is $(\frac{d-1}{2})h$.*
- (b) *If $d - 1$ is odd, then the $\mathbb{A}^1$ -Milnor number is $(\frac{d-2}{2})h + \langle a \rangle$ for some a .*

Proof. We have $f_x = x^{d-1}(da_d + \sum_{i=d+1}^m ia_i x^{i-d})$. Since $da_d \neq 0$, in the local ring $k[x]_{(x)}$, the second factor is a unit. Thus $x^{d-1} = 0$ in $Q_0(\nabla f)$ and $Q_0(\nabla f) \cong k[x]/(x^{d-1})$ and $\mathcal{B} = \{1, x, \dots, x^{d-2}\}$ is a basis. Since $x^k = 0$ in $Q_0(\nabla f)$ for $k > d - 2$, we have $\det H_f = d(d-1)a_d x^{d-2}$ in $Q_0(\nabla f)$. Then η is defined by $\eta(x^{d-2}) = 1/(da_d)$, and $\eta = 0$ on the other basis elements in $\mathcal{B}$. With the ordering given with $\mathcal{B}$, the Gram matrix is an anti-diagonal matrix with the entries $1/(da_d)$ along the diagonal. By Corollary 2.3, the proposition follows. $\square$

As a consequence of Proposition 3.6, the only Grothendieck–Witt classes of ranks 4, 6, and 8 that arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities defined by univariate polynomials are $2h$, $3h$, and $4h$. It remains to consider which forms of ranks 4, 6, and 8 arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities of plane algebraic curves, a question which we study in the subsequent sections of this paper.

4. NEWTON DIAGRAMS, NEWTON NONDEGENERACY, AND KUSHNIRENKO’S BOUND

In this paper, one goal is to determine all Grothendieck–Witt classes of ranks 4, 6, and 8 that arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities for a two-variable polynomial $f: \mathbb{A}^2 \rightarrow \mathbb{A}^1$. To accomplish this, we will use a lower bound for the Milnor number $\mu_0(f)$ given by the Newton number $\nu(f)$ for f . In this section, we provide the background about Newton diagrams, Newton nondegeneracy, and Newton numbers that we need.

First, we recall the definition of a Newton diagram as given in [20]. To a bivariate polynomial $f = \sum_i a_{m_i, n_i} x^{m_i} y^{n_i}$ in the variables x and y , with each $a_{m_i, n_i} \neq 0$, we can associate the collection of points (m_i, n_i) in the plane. The *Newton polyhedron* $\Gamma_+(f)$ of the polynomial f is the convex hull of the regions above and to the right of the marked points (m_i, n_i) . Its boundary is made up of straight line segments or rays. The union of the line segments and those rays not on a coordinate axis, form the *Newton diagram* $\Gamma(f)$ of f . We say that $\Gamma(f)$ and f are *convenient* if $\Gamma(f)$ has a vertex on each coordinate axis. When f is convenient, we define $\Gamma_-(f)$ to be the region enclosed by the coordinate axes and $\Gamma(f)$. We have $\Gamma(f) = \Gamma_+(f) \cap \Gamma_-(f)$.

Let E be an edge of $\nu(f)$. To E , we define an associated polynomial $\phi_E(f) \in k[T]$. We denote the vertices on E as (r_i, s_i) , where

$$(r_i, s_i) = (r_0, s_0) + i(a, b), \quad \text{for } 0 \leq i \leq t,$$

for some relatively prime integers a, b . We define the polynomial $\phi_E(T)$ by $\phi_E(T) = \sum_{i=0}^t a_{r_i, s_i} T^i$ where a_{r_i, s_i} are the (possibly zero) coefficients of f .

A polynomial f is said to be *Newton non-degenerate* if its Newton diagram $\Gamma(f)$ is convenient and, for all edges E of $\Gamma(f)$, the polynomial $\phi_E(T)$ has distinct roots. Note that if there are no interior lattice points on E , then $\phi_E(T) = T^t + a$ for some $t \geq 1$ and $a \neq 0$. As k has characteristic zero, the roots of $\phi_E(t)$ are distinct. Hence, we have the following criterion:

Lemma 4.1. *Assume $\text{char } k = 0$ and f is a convenient polynomial. If the edges of $\Gamma(f)$ do not contain an interior lattice point, then f is Newton non-degenerate.*

We now state a result that polynomials that agree up to terms of a certain order have isometric $\mathbb{A}^1$ -Milnor numbers. Let P_0 be the local ring

$$P_0 = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$$

Lemma 4.2 ([9, 17]). *Let $f: \mathbb{A}^n \rightarrow \mathbb{A}$ be a map with an isolated singularity at the origin and $\mu_0(f) = N$. If the map $g: \mathbb{A}^n \rightarrow \mathbb{A}$ satisfies $f - g \in (x_1, \dots, x_n)^{N+2}$ in P_0 then*

- (a) *The ideals $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ and $(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n})$ are equal in P_0 ;*
- (b) *$Q_0(\nabla f) = Q_0(\nabla g)$; and*
- (c) *$\mu_0^{\mathbb{A}^1}(f) = \mu_0^{\mathbb{A}^1}(g)$.*

Proof. This follows from the proof of [9, Lemma 17], as explained in the proof of [17, Lemma 12]. Since $f - g \in (x_1, \dots, x_n)^{N+2}$, we have that $\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i} \in (x_1, \dots, x_n)^{N+1}$. Then, by [17, Lemma 12], (i) and (ii) are true. The same lemma shows that the EKL formes of $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ and $(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n})$ are equal. As this is the definition of the $\mathbb{A}^1$ -Milnor number, $\mu_0^{\mathbb{A}^1}(f) = \mu_0^{\mathbb{A}^1}(g)$. $\square$

Lemma 4.2 shows that modifying monomial terms in f with total degree at least $N + 2$ changes neither the Milnor algebra $Q_0(\nabla f)$ nor $\mu_0(f)$. Hence, we can safely ignore all monomial terms with total degree at least $N + 2$. Note that since the Milnor number is finite, the modified f still has an isolated singularity. Hence, we only need to consider polynomials f with the following form:

$$f = \sum_{\substack{i,j \\ 1 \leq i+j \leq N+1}} a_{i,j} x^i y^j.$$

If such a polynomial is not convenient, we can use the same lemma to add on x^{N+2} or y^{N+2} without changing the $\mathbb{A}^1$ -Milnor number. Thus it suffices to only consider (convenient) polynomials of the following form:

$$\sum_{\substack{i,j \\ 1 \leq i+j \leq N+1}} a_{i,j} x^i y^j + x^{N+2} + y^{N+2}.$$

Finally, a useful result for our calculations of $\mathbb{A}^1$ -Milnor numbers follows immediately from Quick, Strand, and Wilson [17]. This lemma enables us to start with a finite spanning set to calculate $Q_0(\nabla f)$.

Lemma 4.3. [17, Lemma 10?] *Let $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$ be a map with an isolated zero at 0, and assume $\mu_0(f) = N \geq 1$. Let I be the ideal $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ in $P_0 = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$. Then $(x_1, \dots, x_n)^N \subseteq I$.*

Proof. This follows immediately from [17, Lemma 10] as our $Q_0(\nabla f)$ is the same as their $Q_0(f)$. $\square$

We now explain how to bound the Milnor number of an isolated algebraic singularity in terms of the Newton diagram. Let $f \in k[x, y]$ be a convenient polynomial, let A be the area between the Newton diagram and the axes, and let m (resp. n) is the length of the projection of the Newton diagram to the x (resp. y) axis. When f is convenient, the *Newton number* of f , denoted $\nu(f)$, is given by $\nu(f) = 2A - m - n + 1$. When f is not convenient, we define $\nu(f) := \sup_{m \in \mathbb{N}} \nu(f + x^m + y^n)$.

Kushnirenko showed that the Newton number gives a lower bound on the Milnor number of an isolated algebraic singularity.

Lemma 4.4 (Kushnirenko [11]). *Let k be a field. Assume $f \in k[x, y]$ has an isolated singularity at 0. Then*

$$\mu_0(f) \geq \nu(f).$$

Moreover, this bound is an equality when f is convenient and Newton non-degenerate.

Remark 4.5. Kushnirenko proved this result for formal power series. The Milnor number is an invariant of $Q_0(\nabla f)$, which is the localization of $k[x, y]/(\nabla f)$ and is isomorphic to $k[[x, y]]/(\nabla f)$, the quotient of the formal power series ring as used in [11].

Example 4.6. Let $f = (x + y^2)^2 + xy^3$, as in Example 2.4.

We can use Lemma 4.4 to get a lower bound on $\dim_k Q_0(\nabla f)$. Since $f = (x + y^2)^2 + xy^3 = x^2 + 2xy^2 + xy^3 + y^4$, we plot points $(2, 0)$, $(1, 2)$, $(1, 3)$, and $(0, 4)$ corresponding to the exponents of the monomials in f . The Newton diagram $\Gamma(f)$ of f consists of edge(s) of the convex hull of the four points which is visible from the origin. In this case $\Gamma(f)$ is the line segment connecting $(2, 0)$ and $(0, 4)$.

Here $\nu(f) = 8 - 2 - 4 + 1 = 3$, which is indeed a lower bound for $\mu_0(f) = 4$. The terms of f corresponding to lattice points on the Newton diagram $\Gamma(f)$ are $x^2 + 2xy^2 + y^4 = (x + y)^2$ which has repeated roots. Therefore f is not Newton non-degenerate, contributing to the Newton number $\nu(f) = 3$ falling short of the Milnor number $\mu_0(f) = 4$.

In Section 6, we will use Lemma 4.4 to find all isolated singularities whose Milnor number is at most 8.

5. CHANGE OF VARIABLES

Before proceeding to find all Grothendieck–Witt classes of rank at most 8 that arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities determined by bivariate polynomials, we show in this section that $\mathbb{A}^1$ -Milnor numbers are preserved under certain changes of variables and that, if a polynomial is degenerate, we can often find another polynomial with larger Newton number but the same $\mathbb{A}^1$ -Milnor number.

5.1. $\mathbb{A}^1$ -Milnor number is preserved under invertible change of variables. We define a morphism $\phi: \mathbb{A}^n \rightarrow \mathbb{A}^m$ to be given by $\phi = (\phi_1, \dots, \phi_m)$, with $\phi_i \in k[x_1, \dots, x_n]$. Recall that the Jacobian J_ϕ is the $m \times n$ matrix with $(J_\phi)_{ij} = \frac{\partial \phi_i}{\partial x_j}$. Lemmas 5.1 and 5.2 are well-known, but we provide the proofs for completeness.

Lemma 5.1. *Let $f: \mathbb{A}^n \rightarrow \mathbb{A}$ be a morphism, and let $\phi: \mathbb{A}^n \rightarrow \mathbb{A}^n$ be an invertible morphism. If $g = f \circ \phi$, then*

$$(f_{x_1}(\phi(\mathbf{x})), \dots, f_{x_n}(\phi(\mathbf{x}))) = (g_{x_1}(\mathbf{x}), \dots, g_{x_n}(\mathbf{x}))$$

as ideals in $k[x_1, \dots, x_n]$.

Proof. By the chain rule, $g_{x_i}(\mathbf{x}) = \sum_{j=1}^n f_{x_j}(\phi(\mathbf{x})) \frac{\partial \phi_j}{\partial x_i}(\mathbf{x})$. Thus

$$(g_{x_1}(\mathbf{x}), \dots, g_{x_n}(\mathbf{x})) \subseteq (f_{x_1}(\phi(\mathbf{x})), \dots, f_{x_n}(\phi(\mathbf{x}))).$$

Since ϕ is invertible, we have $f = g \circ \phi^{-1}$. The chain rule gives $(f_{x_1}(\mathbf{x}), \dots, f_{x_n}(\mathbf{x})) \subseteq (g_{x_1}(\phi^{-1}(\mathbf{x})), \dots, g_{x_n}(\phi^{-1}(\mathbf{x})))$. Substituting $\phi(\mathbf{x})$ for $\mathbf{x}$, we obtain the reverse inclusion and the lemma is proved. $\square$

Let $\phi: \mathbb{A}^n \rightarrow \mathbb{A}^n$, $\phi = (\phi_i)$ be an isomorphism. Then ϕ induces a ring isomorphism $\Phi: \mathbb{A}^1 \rightarrow \mathbb{A}^1$, defined by $\Phi(x_i) = \phi_i(\mathbf{x})$. For any $h \in \mathbb{A}^1$, we have $\Phi(h(\mathbf{x})) = h(\phi(\mathbf{x}))$.

Now let $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$ be a morphism with an isolated singularity at the origin. Let $\phi: \mathbb{A}^n \rightarrow \mathbb{A}^n$ be an invertible morphism and let $g = f \circ \phi$. Then by Lemma 5.1, Φ induces a well-defined ring homomorphism $\bar{\Phi}: Q_0(\nabla f) \rightarrow Q_0(\nabla g)$.

Lemma 5.2. *Let $f: \mathbb{A}^n \rightarrow \mathbb{A}$ be a morphism with an isolated singularity at the origin. Let $\phi: \mathbb{A}^n \rightarrow \mathbb{A}^n$ be an invertible morphism, and let $g = f \circ \phi$. Then $\bar{\Phi}: Q_0(\nabla f) \rightarrow Q_0(\nabla g)$ is a ring isomorphism.*

Proof. Let ϕ^{-1} be the inverse to ϕ . Then it defines a ring isomorphism $\Phi^{-1}: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ that is the inverse to Φ . By Lemma 5.1, it induces $\bar{\Phi}^{-1}: Q_0(\nabla g) \rightarrow Q_0(\nabla f)$. Since Φ, Φ^{-1} are inverses, $\bar{\Phi}^{-1} = \bar{\Phi}^{-1}$, and $\bar{\Phi}$ is an isomorphism. $\square$

A corollary of Lemma 5.2 is that for the classical Milnor number, $\mu_0(f) = \mu_0(g)$. We now show that the $\mathbb{A}^1$ -Milnor number is invariant under an invertible change of variables.

Theorem 5.3. *Let $f: \mathbb{A}^n \rightarrow \mathbb{A}$ be a morphism with an isolated singularity at the origin. Let $\phi: \mathbb{A}^n \rightarrow \mathbb{A}^n$ be an invertible morphism with $\phi(0) = 0$ and let $g = f \circ \phi$. Then $Q_0(\nabla f) \cong Q_0(\nabla g)$ and $\mu_0^{\mathbb{A}^1}(g) = \mu_0^{\mathbb{A}^1}(f)$.*

Proof. We will show that $\mu_0^{\mathbb{A}^1}(g) = \mu_0^{\mathbb{A}^1}(f)$, by following the construction of the $\mathbb{A}^1$ -Milnor number and comparing the Hessian H_f , map η_f , and Gram matrix for f with those of g . By the chain rule,

$$H_g(\mathbf{x})_{i,j} = (J_\phi^T(\mathbf{x})H_f(\phi(\mathbf{x}))J_\phi(\mathbf{x}))_{i,j} + \sum_{k=1}^n \frac{\partial f}{\partial x_k}(\phi(\mathbf{x})) \left(\frac{\partial^2 \phi_k}{\partial x_i \partial x_j} \right)(\mathbf{x}).$$

Now $\frac{\partial f}{\partial x_k}(\phi(\mathbf{x})) = f_{x_k}(\phi(\mathbf{x})) = 0$ in $Q_0(\nabla g)$, for all $k \in [n]$. Thus, in $Q_0(\nabla g)$, we have

$$H_g(\mathbf{x}) = J_\phi^T(\mathbf{x})H_f(\phi(\mathbf{x}))J_\phi(\mathbf{x}).$$

Taking determinants,

$$\det H_g(\mathbf{x}) = \det J_\phi(\mathbf{x})^2 \det H_f(\phi(\mathbf{x})).$$

Applying $\bar{\Phi}^{-1}$, we find

$$\det H_g(\phi^{-1}(\mathbf{x})) = \det J_\phi(\phi^{-1}(\mathbf{x}))^2 \det H_f(\mathbf{x}).$$

in $Q_0(\nabla f)$. Since ϕ is invertible, by the chain rule, the matrix J_ϕ is invertible. Then the determinant

$$D = \det J_\phi(\phi^{-1}(\mathbf{x}))$$

is also invertible. Multiplying both sides of the above equation by D^{-2} , we have that

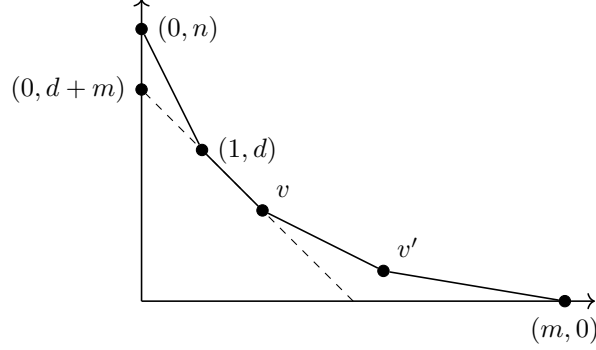
$$D^{-2} \det H_g(\phi^{-1}(\mathbf{x})) = \det H_f(\mathbf{x}).$$

Now suppose we have a k -linear functional $\eta_f: Q_0(\nabla f) \rightarrow k$ with $\eta_f(\det H_f(\mathbf{x})) = \mu_0(f)$. We define a map $\eta_g: Q_0(\nabla g) \rightarrow k$ by

$$\eta_g(m) = \eta_f(D^{-2}\bar{\Phi}^{-1}(m)) \text{ for } m \in Q_0(\nabla g).$$

This is a k -linear map as η_f and $\bar{\Phi}^{-1}$ are k -linear. Now η_g is the desired map needed to define $\mu_0(g)$ as

$$\begin{aligned} \eta_g(\det H_g(\mathbf{x})) &= \eta_f(D^{-2}\bar{\Phi}^{-1}(\det H_g(\mathbf{x}))) \\ &= \eta_f(D^{-2} \det H_g(\phi^{-1}(\mathbf{x}))) \\ &= \eta_f(\det H_f(\mathbf{x})) = \mu_0(f) = \mu_0(g). \end{aligned}$$

FIGURE 1. Possible Newton diagram $\Gamma(f)$ in Lemma 5.5

Using the k -linear maps η_f, η_g , we now show that we can choose bases such that the Gram matrices of $\mu_0^{\mathbb{A}^1}(g)$ and $\mu_0^{\mathbb{A}^1}(f)$ are the same. It then follows that $\mu_0^{\mathbb{A}^1}(f) = \mu_0^{\mathbb{A}^1}(g)$. Let $\mathcal{B} = \{m_\ell(\mathbf{x})\}_{\ell=1}^r$ be a basis of $Q_0(\nabla g)$. Since D is a unit, by Lemma 5.2, it follows that $\mathcal{B}' = \{D^{-1}\Phi^{-1}(m_\ell(\mathbf{x}))\}_{\ell=1}^r$ is a basis of $Q_0(\nabla f)$. Then

$$\begin{aligned} \eta_f(D^{-1}\Phi^{-1}(m_i(\mathbf{x}))D^{-1}\Phi^{-1}(m_j(\mathbf{x}))) &= \eta_f(D^{-2}\Phi^{-1}(m_i(\mathbf{x})m_j(\mathbf{x}))) \\ &= \eta_g(m_i(\mathbf{x})m_j(\mathbf{x})) \end{aligned}$$

for all $i, j \in [r]$. Thus the (i, j) entry of the Gram matrix of $\mu_0^{\mathbb{A}^1}(f)$ with respect to the basis $\mathcal{B}$ is the same as the (i, j) entry of the Gram matrix of $\mu_0^{\mathbb{A}^1}(g)$ with respect to the basis $\mathcal{B}'$. Then $\mu_0^{\mathbb{A}^1}(f) = \mu_0^{\mathbb{A}^1}(g)$. $\square$

5.2. Change of variables and Newton number. In this section, we explicitly show how, in some specific cases, one can use a change of variables to increase the Newton number without changing the $\mathbb{A}^1$ -Milnor number.

Lemma 5.4. *Let $f(x, y) = x^b y^c \in k[x, y]$ for non-negative integers b, c . Define $g(X, Y) \in k[X, Y]$ by $g(X, Y) = f(x, y)$, where $X = x + ay^m$, $Y = y$, for some $m \geq 1$. Then $X^b Y^c$ is a non-zero monomial in $g(X, Y)$ and all the non-zero monomials lie on the line segment from (b, c) to $(0, c + bm)$.*

Proof. As $x^b y^c = (X - aY^m)^b Y^c = \sum_{i=0}^b \binom{b}{i} a^i X^{b-i} Y^{c+mi}$, each monomial $X^s Y^t$ in $g(X, Y)$ has $(s, t) = (b - i, c + im)$ for some $i \geq 0$. $\square$

We recall that if $\Gamma(f)$ is the Newton diagram for f , then $\Gamma_+(f)$ is the set of all lattice points (x, y) lying on or above $\Gamma(f)$, and $\Gamma_-(f)$ is the set of lattice points (x, y) lying on or below $\Gamma(f)$ with $x, y \geq 0$.

Lemma 5.5. *Let $f(x, y) \in k[x, y]$ be convenient. Let e be an edge of $\Gamma(f)$ containing $P = (1, d)$. Assume the slope of e is $\geq -m$, for some positive integer m . Define $g(X, Y) \in k[X, Y]$ by $g(x + ay^m, y) = f(x, y)$. If $(b, c) \in \Gamma_+(g)$, then either*

- (a) (b, c) is in $\Gamma_+(f)$; or
- (b) $b = 0$ and $c \geq d + m$.

Moreover, if $P \in \Gamma(f)$ is a point on an edge e' with slope greater than the slope of e , then $P \in \Gamma(g)$.

Proof. Let $(b, c) \in \Gamma_+(g)$, then by Lemma 5.4, $(b, c) = (r, s) - t(1, -m)$, for some $(r, s) \in \Gamma_+(f)$ and some $0 \leq t \leq r$. If $t = 0$, (1) is proved. Hence $r \geq 1$. Now, by the convexity of the Newton polyhedron $\Gamma_+(f)$, the slopes of the edges are negative and increase as their x -coordinates increase. Hence all the edges in $\Gamma(f)$ with $x \geq 1$ have slopes $\geq -m$. Thus, since $(r, s) \in \Gamma_+(f)$, $(b, c) \in \Gamma_+(f)$ or $b = 0$. If $b = 0$ and $c < d + m$, then the line segment from $(1, d)$ to (r, s) would have slope $< -m$, and thus $(r, s) \notin \Gamma_+(f)$ by definition of $\Gamma(f)$. Hence $c \geq d + m$. Finally, we note that if P is a point on an edge e' with slope greater than that of e , then by Lemma 5.4, $P \in \Gamma(g)$ as there are no points $P + t(1, -m)$ in $\Gamma_+(f)$, with $t > 0$. $\square$

Lemma 5.6. *Let $f, g \in k[x, y]$ be convenient polynomials whose Newton diagrams share the same points on the coordinate axes. If $\Gamma_-(f) \subsetneq \Gamma_-(g)$, then $\nu(g) > \nu(f)$.*

Proof. Let $(m, 0), (0, n)$ be the points on the coordinate axes for $\Gamma(f), \Gamma(g)$. As $\Gamma_-(f) \subsetneq \Gamma_-(g)$, $A_f < A_g$. Then

$$\nu(f) = 2A_f - m - n + 1 < 2A_g - m - n + 1 = \nu(g). \quad \square$$

We now prove the main result of this subsection. For a bivariate polynomial f , we recall that if e is an edge of the Newton diagram $\Gamma(f)$, then f_e is the part of f consisting of the monomials that lie on e .

Theorem 5.7. *Let $f \in k[x, y]$ be convenient and have an isolated singularity at the origin. Let e be a degenerate edge of $\Gamma(f)$ and assume f_e has $x + ay^m$ as a repeated factor. If $P = (1, d)$ is a point on e , then there exists another convenient polynomial $g \in k[x, y]$ with an isolated singularity at the origin such that $\mu_0^{\mathbb{A}^1}(g) = \mu_0^{\mathbb{A}^1}(f)$ and $\nu(g) > \nu(f)$.*

Proof. Let $\phi(x, y) = (x - ay^m, y)$ and let $g = f \circ \phi$. Then by Theorem 5.3, $\mu_0^{\mathbb{A}^1}(g) = \mu_0^{\mathbb{A}^1}(f)$. Let $(n, 0) \in \Gamma(f)$. Then by Lemma 5.4, $(n, 0) \in \Gamma(g)$. By Lemma 4.2, we can add a sufficiently high power of y^N , chosen so that $N > d + m$ to g without changing the $\mathbb{A}^1$ -Milnor number. Hence, we can assume that g is convenient.

We now show that $\nu(g) > \nu(f)$ by creating a polynomial h with the property that $\nu(h) = \nu(f)$ and $\nu(h) < \nu(g)$. First, let M be the smallest positive integer such that the monomial term Y^M in $g(X, Y)$ has a nonzero coefficient. By Lemma 5.4, $M \geq d + m$. We now define a new polynomial $h(x, y)$ obtained from $f(x, y)$ by deleting all the monomial terms y^r in $f(x, y)$, for $r < M$ and letting the coefficient of y^M be 1. Since $(1, k)$ is a point on both $\Gamma(f), \Gamma(h)$, the formula for the Newton number shows that $\nu(h), \nu(f)$ are independent of the choice of point $(0, r)$ on their Newton diagram. Hence, $\nu(f) = \nu(h)$.

We now show $\nu(h) < \nu(g)$ to prove the theorem. By assumption $f_e(x, y)$ is a multiple of $(x + ay^m)^s = X^s$, for some $s \geq 2$. By Lemma 5.4, all vertices in $\Gamma(g)$ lying on the extension of e to the y -axis must come from a non-zero monomial $X^u Y^v$ in $g(X, Y) = f_e(X - aY^m, Y)$, which is a multiple of X^s . Hence neither $(1, d)$ nor $(0, d + m)$ is in $\Gamma(g)$. As a consequence, $\Gamma_-(h) \subsetneq \Gamma_-(g)$. As both g, h are convenient and their Newton diagrams contain the same points $(n, 0), (0, M)$ on the coordinate axes, by Lemma 5.6, $\nu(g) > \nu(h)$. This proves the theorem. $\square$

When the horizontal change of the degenerate edge is at most 3, the conclusion of the theorem applies.

Corollary 5.8. *Let f be a convenient bivariate polynomial with isolated singularity at the origin. Assume that e is a degenerate edge in $\Gamma(f)$ and the endpoints of e are given by*

$(a_0, b_0), (a_1, b_1)$ with $a_1 > a_0$. If $a_1 - a_0 \leq 3$, then exists a convenient polynomial g with an isolated singularity at the origin such that $\mu_0^{\mathbb{A}^1}(g) = \mu_0^{\mathbb{A}^1}(f)$ and $\nu(g) > \nu(f)$.

Proof. The endpoint condition on e ensures that the polynomial $f_e(x, y)$ is of the form $x^{a_0}y^{b_1}g(x, y)$, where $g(x, y) = x^{a_1-a_0} + \dots + y^{b_0-b_1}$. The degeneracy condition shows that $g(x, y)$ has a repeated factor of the form $x^s + ay^m$. Since $\deg g \leq 3$, the repeated factor is of the form $x + ay^m$. Hence Theorem 5.7 can be applied to increase the Newton number. $\square$

Example 5.9. Let $f = (x + y^2)^2 + xy^3$. In Example 2.4, we calculated $\mu_0^{\mathbb{A}^1}(f) = 2h$. Now f is Newton degenerate as its Newton diagram is the edge E from $(0, 4)$ to $(2, 0)$ with interior vertex $(1, 2)$. As the associated polynomial $\phi_E(T) = (T + 1)^2$ has repeated roots, f is Newton degenerate. The Newton number $\nu(f)$ is $2(4) - 4 - 2 + 1 = 3$ and one has the strict inequality $\mu_0(f) > \nu(f)$.

We now show how to apply a linear change of variables to transform f into another polynomial g which is Newton non-degenerate but has the same $\mathbb{A}^1$ -Milnor number. Kushnirenko's theorem applied to g will then give $\mu_0(g) = \nu(g)$.

After the change of variables $x_1 = x + y^2$ and $y_1 = y$, we obtain

$$g(x_1, y_1) = x_1^2 + x_1y_1^3 - y_1^5 = f(x, y)$$

By Theorem 5.3, $\mu_0^{\mathbb{A}^1}(f) = \mu_0^{\mathbb{A}^1}(g)$. Moreover, as the Newton diagram of g is the edge from $(2, 0)$ to $(0, 5)$, it is convenient and Newton non-degenerate. Kushnirenko's Theorem 4.4 shows that $\mu_0(g) = 10 - 2 - 5 + 1 = 4$. Hence $\mu_0(f) = 4$, as calculated above.

6. CLASSIFICATION OF NEWTON DIAGRAMS WITH NEWTON NUMBER AT MOST 8

Let f be a bivariate convenient polynomial with an isolated singularity at 0. In this section, we determine all such f with Newton number $\nu(f) \leq 8$. Because $\mu_0(f) \geq \nu(f)$ by Lemma 4.4, we thus obtain a lower bound on $\mu_0(f)$. In Section 7, we use this bound to classify all Grothendieck–Witt classes of rank at most 8 that can arise as $\mathbb{A}^1$ -Milnor number of bivariate polynomial f .

Let f be a bivariate convenient polynomial in the variables x and y . Let m (resp. n) be the smallest integer for which the coefficient of x^m (resp. y^n) in f is nonzero. Recall that $A(f)$ is the area between the Newton diagram $\Gamma(f)$ and the axes, and

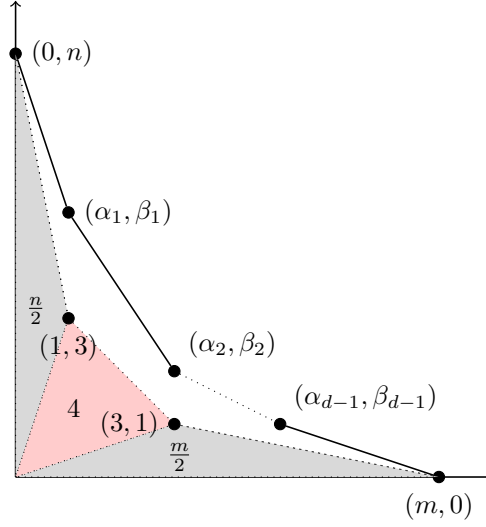
$$\nu(f) = 2A(f) - m - n + 1.$$

Let d be the number of edges (with different slopes) in $\Gamma(f)$. An edge comprised of multiple line segments is considered a single edge.

As we are focused on polynomials with Milnor number at most 8, the following lemma allows us to restrict attention to cases where $\Gamma(f)$ has at most three edges.

Lemma 6.1. *Let f be a bivariate convenient polynomial with an isolated singularity at 0. Let $\Gamma(f)$ have d edges. If $d \geq 4$, then $\nu(f) \geq 9$.*

Proof. Suppose $d \geq 4$, and let $(0, n), (\alpha_1, \beta_1), \dots, (\alpha_{d-1}, \beta_{d-1})$, and $(m, 0)$, be the vertices at the ends of the d edges, as shown in Figure 2. We have $\alpha_1, \beta_{d-1} \geq 1$, and moreover $\alpha_i, \beta_{d-i} \geq i$ for all $i \in [d-1]$. Thus $\alpha_i + \beta_i \geq d \geq 4$ for all $i \in [d-1]$. First we claim and justify that the shaded region in Figure 2 is contained in the region between $\Gamma(f)$ and the axes: From a point (i, j) between adjacent vertices on the Newton diagram $\Gamma(f)$, if we move in the southeast (respectively northwest) direction, then the y -coordinate (respectively x -coordinate) must decrease by at least 1. This implies the maximum number of edges in $\Gamma(f)$ which contains a vertex with coordinate (i, j) is $i + j$. So the Newton diagram $\Gamma(f)$

FIGURE 2. Case when $\Gamma(f)$ has $d \geq 4$ edges.

with four edges can only contain vertices (i, j) with $i + j \geq 4$. This means the vertices of $\Gamma(f)$ are outside the triangle with vertices at $(0, 0)$, $(3, 0)$, and $(0, 3)$. Since $(m, 0)$ is the vertex of $\Gamma(f)$ on the x -axis which is closest to the origin, there is no vertex of $\Gamma(f)$ along the x -axis to the left of $(m, 0)$. Therefore the triangle with vertices at $(4, 0)$, $(3, 1)$, and $(m, 0)$ is contained within the region in the first quadrant between $\Gamma(f)$ and the two axes. Similarly, we can conclude that the triangle with vertices at $(0, 4)$, $(1, 3)$, and $(0, n)$ is also within the region and complete the argument for the claim. As the area of the shaded region is $4 + \frac{m+n}{2}$, we have $A(f) \geq 4 + \frac{m+n}{2}$. Hence $\nu(f) = 2A(f) - m - n + 1 \geq 8 + 1 = 9$. $\square$

As a result of Lemma 6.1, it suffices to consider the cases when $\Gamma(f)$ has one, two, or three edges. The next lemma gives the possibilities when $\Gamma(f)$ has one edge.

Lemma 6.2. *Let f be a bivariate, convenient polynomial with an isolated singularity at 0, and $\nu(f) \leq 8$. If $\Gamma(f)$ consists of a single edge, then the edge has endpoints $\{(m, 0), (n, 0)\}$, or by symmetry $\{(n, 0), (m, 0)\}$, with (m, n) given by*

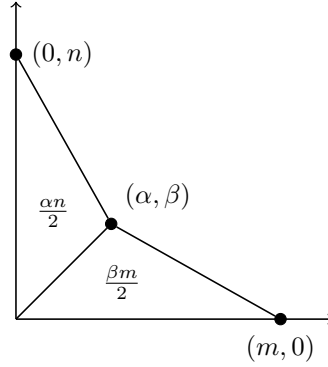
$\nu(f)$	1	2	3	4	5	6	7	8
(m, n)	(2, 2)	(2, 3)	(2, 4)	(2, 5), (3, 3)	(2, 6)	(2, 7), (3, 4)	(2, 8)	(2, 9), (3, 5)

Proof. Let the edge of $\Gamma(f)$ have terminal vertices $(0, n)$, $(m, 0)$. Then $A(f) = nm/2$ and $\nu(f) = (m-1)(n-1)$. By symmetry, assume that $m \leq n$. For $\nu(f) \leq 8$, the possible (m, n) are given by the table. $\square$

The next lemma gives the possibilities when $\Gamma(f)$ has two edges.

Lemma 6.3. *Let f be a bivariate, convenient polynomial with an isolated singularity at 0 and $\nu(f) \leq 8$. If $\Gamma(f)$ consists of two edges, then the edges have endpoints $\{(0, n), (\alpha, \beta), (m, 0)\}$, or by symmetry $\{(n, 0), (\beta, \alpha), (0, m)\}$, with α, β, m, n as follows:*

$\nu(f)$	Possible α, β, m, n
1	$(\alpha, \beta) = (1, 1)$, with $m, n \geq 2$
2	$\emptyset$
3	$(\alpha, \beta) = (2, 1)$, with $m \geq 5$ and $n = 2$
4	$(\alpha, \beta) = (2, 1)$, with $m \geq 4$ and $n = 3$
5	$(\alpha, \beta) = (2, 1)$, with $m \geq 3$ and $n = 4$ $(\alpha, \beta) = (3, 1)$, with $m \geq 7$ and $n = 2$
6	$(\alpha, \beta) = (2, 1)$, with $m \geq 3$ and $n = 5$
7	$(\alpha, \beta) = (2, 1)$, with $m \geq 3$ and $n = 6$ $(\alpha, \beta) = (3, 1)$, with $m \geq 5$ and $n = 3$ $(\alpha, \beta) = (4, 1)$, with $m \geq 9$ and $n = 2$
8	$(\alpha, \beta) = (2, 1)$, with $m \geq 3$ and $n = 7$

FIGURE 3. $\Gamma(f)$ with two edges

Proof. Let the terminal vertices of the two edges of $\Gamma(f)$ be $(0, n)$, (α, β) , $(m, 0)$, where $0 < \beta < n$ and $0 < \alpha < m$. Computing areas as shown in Figure 3, we have that

$$\nu(f) = m(\beta - 1) + n(\alpha - 1) + 1.$$

For each value of $\nu(f) \leq 8$, we determine the corresponding Newton diagrams. If $\alpha, \beta \geq 2$, then $m, n \geq 3$, and moreover either $m \geq 5$ or $n \geq 5$ in order for (α, β) to be a vertex of the Newton diagram. Thus we would have that $\nu(f) = m(\beta - 1) + n(\alpha - 1) + 1 \geq 9$. By symmetry, assume $\alpha \geq \beta$. In finding polynomials with $\nu(f) \leq 8$, we can therefore assume $\beta = 1$, in which case $\nu(f) = n(\alpha - 1) + 1$. As the right side of this equation is positive, $\nu(f) = 0$ is not possible.

- $\nu(f) = 1$: We must have $\alpha = 1$, in which case there are no further restrictions on m and n .
- $\nu(f) = 2$: Then $n(\alpha - 1) = 1$. As $n \geq 2$, there are no solutions.
- $\nu(f) = 3$: Then $n(\alpha - 1) = 2$. As $n \geq 2$, we must have $\alpha = 2$ and $n = 2$. This gives two edges exactly when $m \geq 5$.
- $\nu(f) = 4$: Then $n(\alpha - 1) = 3$. As $n \geq 2$, we must have $\alpha = 2$ and $n = 3$. This gives two edges exactly when $m \geq 4$.
- $\nu(f) = 5$: Then $n(\alpha - 1) = 4$, which forces either $(n, \alpha) = (4, 2)$ or $(n, \alpha) = (2, 3)$. In the first case, this gives two edges exactly when $m \geq 3$. In the second case, it gives two edges exactly when $m \geq 7$.

- $\nu(f) = 6$: Then $n(\alpha - 1) = 5$, which forces $(n, \alpha) = (5, 2)$. This gives two edges exactly when $m \geq 3$.
- $\nu(f) = 7$: Then $n(\alpha - 1) = 6$, which forces either $(n, \alpha) = (6, 2)$, $(n, \alpha) = (3, 3)$, or $(n, \alpha) = (2, 4)$. In the first case, this gives two edges exactly when $m \geq 3$. In the second case, it gives two edges exactly when $m \geq 5$. In the third case, it gives two edges exactly when $m \geq 9$.
- $\nu(f) = 8$: Then $n(\alpha - 1) = 7$, which forces $(n, \alpha) = (7, 2)$. This gives two edges when $m \geq 3$. $\square$

The following lemma gives the possibilities when $\Gamma(f)$ has three edges.

Lemma 6.4. *Let f be a convenient bivariate polynomial with an isolated singularity at 0 and $\nu(f) \leq 8$. If $\Gamma(f)$ consists of three edges, then $\nu(f)$ is 4, 6, or 8. Moreover, the edges of $\Gamma(f)$ have terminal vertices $\{(0, n), (\alpha, \beta), (\gamma, \delta), (m, 0)\}$, or by symmetry $\{(0, m), (\delta, \gamma), (\beta, \alpha), (n, 0)\}$, with $\gamma, \delta, \alpha, \beta, m, n$ as follows:*

$\nu(f)$	Possible $\gamma, \delta, \alpha, \beta, m, n$
4	$(\gamma, \delta) = (2, 1)$ and $(\alpha, \beta) = (1, 2)$, with $m, n \geq 4$
6	$(\gamma, \delta) = (2, 1)$ and $(\alpha, \beta) = (1, 3)$, with $m \geq 3$ and $n \geq 6$
8	$(\gamma, \delta) = (2, 1)$ and $(\alpha, \beta) = (1, 4)$, with $m \geq 3$ and $n \geq 8$.

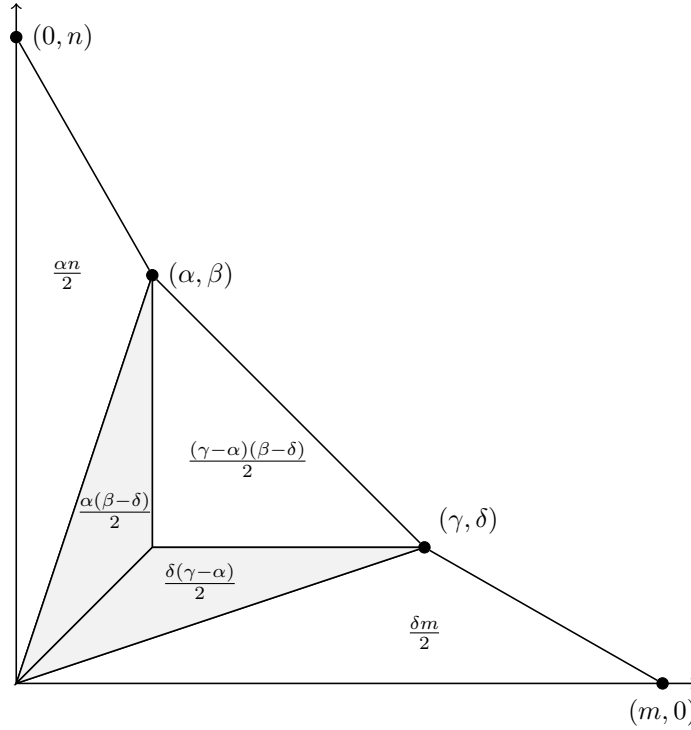


FIGURE 4. $\Gamma(f)$ with three edges

Proof. Let the three edges of $\Gamma(f)$ have terminal vertices $\{(0, n), (\alpha, \beta), (\gamma, \delta), (m, 0)\}$, where $0 < \delta < \beta < n$ and $0 < \alpha < \gamma < m$. Computing areas as shown in Figure 4, we have that

$$(2) \quad \nu(f) = n(\alpha - 1) + m(\delta - 1) + \alpha(\beta - \delta) + \delta(\gamma - \alpha) + (\gamma - \alpha)(\beta - \delta) + 1.$$

If $\delta \geq 2$, this quantity can be at most 8 only if $\alpha = 1$, $m = 3$, $\beta = 3$, and $\gamma = 2$. However, this does not give a Newton diagram with three edges, so we conclude that we must have $\delta = 1$. Similarly, $\alpha = 1$.

Under the assumption $\delta = \alpha = 1$, (2) simplifies to $\nu(f) = \gamma\beta$. By symmetry, suppose $\gamma \leq \beta$. As $\gamma, \beta \geq 2$, the only possibilities are $\nu(f) = 4$ with $\gamma = 2$ and $\beta = 2$, $\nu(f) = 6$ with $\gamma = 2$ and $\beta = 3$, and $\nu(f) = 8$ with $\gamma = 2$ and $\beta = 4$. In the first of these cases, we have three edges exactly when $m, n \geq 4$. In the second case, we have three edges exactly when $m \geq 3$ and $n \geq 6$. In the third case, we have three edges exactly when $m \geq 3$ and $n \geq 8$. $\square$

Having found all Newton diagrams with Newton number at most 8, we will use them in the next section to find all Grothendieck–Witt classes of rank at most 8 that arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities of plane algebraic curves.

7. $\mathbb{A}^1$ -MILNOR NUMBERS OF RANK AT MOST 8 FOR BIVARIATE f

- Important: Show that Kushnirenko’s definition of non-degenerate implies $\Delta \neq 0$. Should be put in Section 4, where we introduce non-degeneracy.
- ”Row reduction (2)” needs to specify the (2) in here? (done?)

Let $f \in k[x, y]$ be a convenient, bivariate polynomial with an isolated singularity at zero. In this section, we determine the $\mathbb{A}^1$ -Milnor numbers $\mu_0^{\mathbb{A}^1}(f)$ that can occur when $\mu_0(f) \leq 8$. By Theorem 3.5, all $\mathbb{A}^1$ -Milnor numbers of ranks 1, 2, 3, 5, and 7 arise from a single variable function f . Thus, we only need to consider the cases when $\mu_0(f) = 4, 6, 8$. In subsection 7.1, we show that we can assume f is Newton nondegenerate. Then in section 7.2, we classify all $\mathbb{A}^1$ -Milnor numbers $\mu_0^{\mathbb{A}^1}(f)$ that can occur for ranks 4, 6, 8 for bivariate f .

7.1. Reduction to Newton nondegenerate cases. The following theorem allows us to restrict attention to Newton nondegenerate polynomials.

Theorem 7.1. *Let f be a bivariate polynomial that is convenient but not Newton non-degenerate. If $\nu(f) \leq 8$, then there is a bivariate convenient polynomial g such that $\mu_0^{\mathbb{A}^1}(g) = \mu_0^{\mathbb{A}^1}(f)$ and $\nu(g) > \nu(f)$.*

Proof. If a bivariate polynomial is convenient with Newton number at most 8, then it is of one of the forms listed above. If it is Newton degenerate, by interchanging variables if necessary, it has a degeneracy along an edge of the Newton diagram that passes through or is incident to a point of the form $(1, k)$. Moreover, in each case the repeated factor must be of the form $x + ay^\alpha$. Applying Theorem 5.7 then gives the result. $\square$

Corollary 7.2. *Let f be a bivariate convenient polynomial with an isolated singularity at zero. If f is Newton-degenerate, then there exists a Newton-nondegenerate bivariate convenient polynomial g with an isolated singularity at zero, and such that $\mu_0^{\mathbb{A}^1}(f) = \mu_0^{\mathbb{A}^1}(g)$.*

Proof. By Theorem 7.1, there exists a polynomial f_1 with the same properties as f and $\nu(f_1) > \nu(f)$. If f_1 is Newton non-degenerate, then $g = f_1$ is the desired polynomial. If f_1 is degenerate, then continuing this process, we obtain a sequence $f_1, f_2, \dots, f_\ell$ of

polynomials, with the same properties as f , where $f_1, \dots, f_{\ell-1}$ are Newton-degenerate and $\nu(f) < \nu(f_1) < \dots < \nu(f_\ell)$. If f_ℓ is Newton non-degenerate, then $g = f_\ell$ proves the corollary. for $\ell = \mu_0(f) - \nu(f) + 1$, we have

$$\nu(f_\ell) \geq \nu(f) + \ell = \mu_0(f) + 1 = \mu_0(f_\ell) + 1$$

which contradicts $\mu_0(f_\ell) \geq \nu(f_\ell)$. $\square$

Corollary 7.2 shows that to find all $\mathbb{A}^1$ -Milnor numbers of rank at most 8 coming from a bivariate polynomial f , it suffices to consider Newton nondegenerate polynomials with rank at most 8. By Theorem 3.5, it remains to determine the $\mathbb{A}^1$ -Milnor numbers of ranks 4, 6, 8 that can occur for bivariate polynomials. In the following subsections, we determine the $\mathbb{A}^1$ -Milnor numbers that can occur for these ranks when f is bivariate.

7.2. Classification of $\mathbb{A}^1$ -Milnor numbers in Ranks 4, 6, 8. In this subsection, we classify the $\mathbb{A}^1$ -Milnor numbers $\mu_0^{\mathbb{A}^1}(f)$ that arise for a bivariate polynomial $f: \mathbb{A}^2 \rightarrow \mathbb{A}^2$ with an isolated zero. By Corollary 7.2 and Lemma 4.2, we can assume f is Newton nondegenerate and convenient.

In each case, we apply the following approach to compute $\mu_0(f)$. Suppose f has Milnor number $\mu_0(f) = n$. Since f is convenient and nondegenerate, $\nu(f) = n$ by Lemma 4.4. Then from Section 6, we know the possible Newton diagrams of f with $\nu(f) = n$, and thus the possible monomial terms in f . We then compute a basis of $Q_0(\nabla f)$ and express all monomials in terms of this basis. Using this information, we can calculate the $\mathbb{A}^1$ -Milnor numbers $\mu_0^{\mathbb{A}^1}(f)$ that can arise.

To calculate $Q_0(\nabla f)$, we use the following method. By Lemma 4.3, if $\mu_0(f) = n$, then $Q_0(\nabla f)$ is spanned by the monomials in the set

$$S = \{\text{monomials } x^i y^j \text{ with } 0 \leq i + j < n\}$$

As the partial derivatives f_x, f_y generate the Jacobian ideal and are 0 in $Q_0(\nabla f)$, we have the equations $f_x = 0, f_y = 0$ in $Q_0(\nabla f)$. Similarly, we have equations

$$(3) \quad \alpha f_x = 0, \quad \alpha f_y = 0$$

for all $\alpha \in S$. We regard these equations as a system of linear equations in $Q_0(\nabla f)$, in the monomials in S , and with coefficients in k . Row-reduction of this system gives a basis of $Q_0(\nabla f)$ as a vector space over k and expressions of all $\alpha \in S$ in terms of the basis. We then use this information to compute $\mu_0^{\mathbb{A}^1}(f)$. Before doing the general cases, we illustrate the method with an explicit example.

Example 7.3. We continue Example 5.9. Let $g(x, y) = x^2 + xy^3 - y^5$. The Newton diagram consists of the edge from $(2, 0)$ to $(0, 5)$. It is convenient and non-degenerate. As $\mu_0(g) = 4$, we know that $Q_0 = Q_0(\nabla g)$ is spanned by monomials with degree ≤ 3 , namely the 10 monomials in

$$S = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}.$$

Now $g_x = 2x + y^3$ and $g_y = 3xy^2 - 5y^4$ are both 0 in Q_0 . Similarly, $\alpha g_x = 0, \alpha g_y = 0$, for all $\alpha \in S$. The equation $xg_x = 0$ gives the equation $2x^2 + xy^3 = 0$. Since it has degree 4, $xy^3 = 0$ in Q_0 and thus $x^2 = 0$. There are a total of 20 equations in 10 unknowns (corresponding to the elements of S). Solving this system of equations shows that

$$\{1, y, y^2, y^3\} \text{ is a basis of } Q_0.$$

Row-reduction shows that $x = -y^3/2$, and that x^2, xy, x^3, x^2y, xy^2 are zero in Q_0 .

Then the Hessian determinant $\det H_g = 12xy - 40y^3 - 9y^4 = -40y^3$ in Q_0 . We define $\eta(y^3) = -1/10$ so that $\eta(\det H_g) = 4 = \mu_0(g)$ and $\eta = 0$ on the other basis elements. Finally, the Gram matrix is an anti-diagonal matrix with $\eta(y^3) = -1/10$ along the anti-diagonal. Hence, $\mu_0^{\mathbb{A}^1}(g) = 2h$.

7.3. Rank 4. By the results of Section 6, $\nu(f) = 4$ can be divided into four cases: (i) $\Gamma(f)$ is the line with vertices $(2, 0)$ to $(0, 5)$; (ii*a*) $\Gamma(f)$ is a line from $(3, 0)$ to $(0, 3)$ with no internal vertices; (ii*b*) $\Gamma(f)$ is a line from $(3, 0)$ to $(0, 3)$ with internal vertices; and (ii*i*) $\Gamma(f)$ has two or three edges.

Case (i). As $\Gamma(f)$ has vertices $(2, 0)$ and $(0, 5)$, we can assume

$$f = a_{2,0}x^2 + a_{0,5}y^5 + \sum_{\substack{i,j \\ 3 \leq i+j \leq 5}} a_{i,j}x^i y^j$$

where $a_{0,3} = a_{0,4} = a_{1,2} = 0$ and $a_{2,0}, a_{0,5} \neq 0$. We obtain

- $1, y, y^2, y^3$, is a k -basis of $Q_0(\nabla f)$.
- x is a k -multiple of y^3
- $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

A direct calculation of $\det H_f$ using these results, gives, $\det H_f = 40a_{2,0}a_{0,5}y^3 \neq 0$, when $\text{char } k \neq 2, 5$. We then define η by $\eta(y^3) = a$, where $a = 1/(10a_{2,0}a_{0,5})$ and $\eta(z) = 0$ on all other basis elements z . Then $\eta(\det H_f) = 4$, the Milnor rank.

The Gram matrix is then the antidiagonal matrix

$$\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$$

and $\mu_0^{\mathbb{A}^1}(f) = 2h$ by Corollary 2.3.

*Case (ii*a*).* By assumption on $\Gamma(f)$, we can assume f has the form

$$f = a_{3,0}x^3 + a_{0,3}y^3 + \sum_{\substack{i,j \\ 4 \leq i+j \leq 5}} a_{i,j}x^i y^j$$

where $a_{3,0}, a_{0,3} \neq 0$. We obtain

- $1, x, y, xy$ is a k -basis of $Q_0(\nabla f)$.
- $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

A direct calculation of $\det H_f$ using these results, gives, $\det H_f = 36a_{3,0}a_{0,3}xy \neq 0$, when $\text{char } k \neq 2, 3$. We then define η by $\eta(xy) = a$, where $a = 1/(9a_{3,0}a_{0,3})$, and $\eta = 0$ on all other basis elements. Then $\eta(\det H_f) = 4$, the Milnor number. The Gram matrix is then the antidiagonal matrix

$$\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix},$$

and $\mu_0^{\mathbb{A}^1}(f) = 2h$ by Corollary 2.3.

Case (iib). Here we assume that $a_{3,0} \neq 0$, $a_{0,3} \neq 0$, and either $a_{2,1}$ or $a_{1,2}$ is nonzero. Let

$$g = a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3$$

the smallest degree terms of f . As we assume f is Newton nondegenerate, g has no repeated roots and $\Delta_g \neq 0$, where Δ_g is the discriminant of g given by

$$\Delta_g = -27a_{3,0}^2a_{0,3}^2 + 18a_{3,0}a_{2,1}a_{1,2}a_{0,3} - 4a_{2,1}^3a_{0,3} - 4a_{3,0}a_{1,2}^3 + a_{2,1}^2a_{1,2}^2$$

Let $A = 6a_{3,0}a_{1,2} - 2a_{2,1}^2$, $B = 6a_{2,1}a_{0,3} - 2a_{1,2}^2$, and $C = a_{2,1}a_{1,2} - 9a_{3,0}a_{0,3}$. Algebraic manipulation gives:

$$\Delta_g = \begin{cases} \frac{-1}{12a_{2,1}^2} (a_{1,2}^2A^2 + A^2B - 2a_{2,1}^2AB + 9a_{3,0}^2B^2) & \text{if } a_{2,1} \neq 0 \\ \frac{-1}{12a_{1,2}^2} (a_{2,1}^2B^2 + B^2A - 2a_{1,2}^2AB + 9a_{0,3}^2A^2) & \text{if } a_{1,2} \neq 0. \end{cases}$$

Since either $a_{2,1}$ or $a_{1,2}$ is nonzero, we have that $A \neq 0$ or $B \neq 0$. By symmetry, we can assume that $A \neq 0$. In this case, we obtain

- (a) $1, x, y$, and y^2 is a k -basis of $Q_0(\nabla f)$.
- (b) $x^2 = \frac{B}{A}y^2$ and $xy = \frac{C}{A}y^2$
- (c) $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

A direct calculation of $\det H_f$ using these results gives $\det H_f = \frac{12\Delta_g}{A}y^2 \neq 0$, when $\text{char } k \neq 2, 3$. We then define η by $\eta(y^2) = \frac{A}{3\Delta_g}$, and $\eta = 0$ on all other basis elements. Then $\eta(\det H_f) = 4$, the Milnor number.

Using the ordering $\{1, y, x, y^2\}$, the corresponding Gram matrix is

$$\begin{bmatrix} 0 & 0 & 0 & \frac{A}{3\Delta_g} \\ 0 & \frac{A}{3\Delta_g} & \frac{C}{3\Delta_g} & 0 \\ 0 & \frac{C}{3\Delta_g} & \frac{B}{3\Delta_g} & 0 \\ \frac{A}{3\Delta_g} & 0 & 0 & 0 \end{bmatrix}.$$

The determinant of the middle 2×2 block is $1/3\Delta_g$ and the corresponding symmetric bilinear form is equivalent to $\langle A/3\Delta_g, 1/A \rangle \cong \langle 3\Delta_g A, A \rangle$. Therefore the entire form decomposes as

$$h + \langle 3\Delta_g A, A \rangle.$$

Choosing $a_{2,1} = 0$, $a_{1,2} = 2/5$, $a_{0,3} = 2/25$, $a_{3,0} = 5/6$, we obtain $A = 2$, $3\Delta_g A = -2$, and $\mu_0^{\mathbb{A}^1}(f) = 2h$. Hence, we also obtain the $\mathbb{A}^1$ -Milnor numbers previous found in cases (i), (iia).

Case (iii). We assume that $\Gamma(f)$ has two or three edges. From Lemmas 6.3 and 6.4, we can assume that f has the form

$$(4) \quad f = \sum_{\substack{i,j \\ 3 \leq i+j \leq 5}} a_{i,j} x^i y^j$$

where

- $a_{3,0} = 0$,
- $a_{2,1} \neq 0$, and
- $a_{1,2} \neq 0$ or $a_{0,3} \neq 0$.

Since it suffices to consider only Newton nondegenerate cases, we can assume $D \neq 0$, where

$$D = a_{1,2}^2 - 4a_{2,1}a_{0,3}.$$

Row-reduction of the equations coming from (3) (using only the assumptions $D \neq 0$ and those following (4) above), we obtain

- $1, x, y, y^2$ form a k -basis of $Q_0(\nabla f)$.
- $x^2 = \frac{1}{a_{2,1}^2}(-3a_{2,1}a_{0,3} + a_{1,2}^2)y^2$
- $xy = -\frac{a_{1,2}}{2a_{2,1}}y^2$
- $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

Applying these results, a direct calculation of $\det H_f$ shows that it has only three nonzero terms in $Q_0(\nabla f)$, and they simplify to $\det H_f = -6Dy^2 \neq 0$, when $\text{char } k \neq 2, 3$. We can then define η by $\eta(y^2) = a$, where $a = -\frac{2}{3D}$, and $\eta = 0$ on all other basis elements. Then $\eta(\det H_f) = 4$, the Milnor number.

Using the ordering $\{1, y, x, y^2\}$, the corresponding Gram matrix is the block upper anti-triangular 4×4 matrix:

$$\begin{pmatrix} * & & & a \\ & B & & \\ & & a & \\ a & & & 0 \end{pmatrix}$$

with

$$B = a \begin{bmatrix} 1 & -b/2 \\ -b/2 & b^2 - 3\frac{a_{0,3}}{a_{2,1}} \end{bmatrix}, \quad \text{where } b = \frac{a_{1,2}}{a_{2,1}}.$$

Using Lemma 2.2, $\mu_0^{\mathbb{A}^1}(f) = h + q_B$, where q_B is the form corresponding to B . Since the form q_B represents a , $q_B \cong \langle a, \frac{\det B}{a} \rangle$. As $\det B/a$ simplifies to $-\frac{1}{2a_{2,1}^2}$,

$$q_B \cong \langle -\frac{2}{3D}, -\frac{1}{2a_{2,1}^2} \rangle \cong \langle -2, -6D \rangle$$

Thus

$$\mu^{\mathbb{A}^1}(f) \cong h + \langle -2, -6D \rangle$$

By choosing the parameters $a_{2,1} = 1/24$, $a_{1,2} = 0$, and $a_{0,3} = c$, we have $-6D = c$. Hence, the symmetric bilinear forms that arise as $\mathbb{A}^1$ -Milnor numbers of rank 4 in case (iii) are exactly those of the form $h + \langle -2, c \rangle$ for any nonzero c .

In conclusion, we have found that the $\mathbb{A}^1$ -Milnor numbers $\mu_0^{\mathbb{A}^1}(f)$ that can occur are:

$$\text{Cases (i), (iia): } 2h \quad \text{Case (iib): } h + \langle 3\Delta_g A, A \rangle \quad \text{Case (iii): } h + \langle -2, c \rangle$$

By relaxing the requirement $a_{3,0} \neq 0$ in case (iib) and letting $a_{3,0} = 0$, $a_{2,1} = 1$, $a_{1,2} = 0$, and $a_{0,3} = c/24$, the form in case (iib) becomes $h + \langle -2, c \rangle$. Moreover, letting $c = 2$, this gives $2h$. Thus, all $\mathbb{A}^1$ -Milnor numbers of rank 4 are of the form $h + \langle 3\Delta_g A, A \rangle$, where we recall that $A = 6a_{3,0}a_{1,2} - 2a_{2,1}^2$, Δ_g , and the coefficients $a_{i,j}$ satisfy the conditions in Case (iib) or the relaxed conditions specified above.

7.4. Rank 6. We now assume $\nu(f) = 6$. From Section 6, there are three cases: (i) $\Gamma(f)$ has one edge and consists of $(2, 0)$, $(0, 7)$; (ii) $\Gamma(f)$ has one edge and consists of $(3, 0)$, $(0, 4)$; and (iii) $\Gamma(f)$ has two or three edges.

Case (i). As $\Gamma(f)$ has one edge with vertices $(2, 0)$, $(0, 7)$, we can assume f has the form

$$\sum_{\substack{i,j \\ 2 \leq i+j \leq 7}} a_{i,j} x^i y^j$$

where $a_{1,1} = a_{1,2} = a_{1,3} = a_{0,2} = a_{0,3} = a_{0,4} = a_{0,5} = a_{0,6} = 0$ and $a_{2,0}$ and $a_{0,7}$ are nonzero. Row-reduction of equations as described in (3) in Subsection 7.2 shows

- $1, y, y^2, y^3, y^4, y^5$ form a basis of $Q_0(\nabla f)$.
- x is a k -linear combination of y^4, y^5
- xy is a k -multiple of y^5
- $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

A calculation of $\det H_f$ in $Q_0(\nabla f)$, using these results to reduce terms, shows

$$\det H_f = 84a_{2,0}a_{0,7}y^5,$$

which is non-zero if $\text{char } k \neq 2, 3, 7$. We then define $\eta: Q_0(\nabla f) \rightarrow k$ by $\eta(y^5) = a$, where $a = 1/14a_{2,0}a_{0,7}$, and $\eta = 0$ on all other basis elements. Then $\eta(\det H_f) = 6$, the Milnor number. Direct calculation of the Gram matrix gives the antidiagonal 6×6 matrix

$$\begin{bmatrix} 0 & \cdots & 0 & a \\ \vdots & \ddots & a & 0 \\ 0 & a & \ddots & \vdots \\ a & 0 & \cdots & 0 \end{bmatrix},$$

By Corollary 2.3, this shows $\mu_0^{\mathbb{A}^1}(f) = 3h$.

Case (ii). As $\Gamma(f)$ has one edge with vertices $(3, 0), (0, 4)$, we can assume f has the form

$$\sum_{\substack{i,j \\ 3 \leq i+j \leq 7}} a_{i,j} x^i y^j$$

where $a_{1,2} = a_{0,3} = a_{2,1} = 0$ and $a_{3,0}, a_{0,4}$ are nonzero. Row-reduction of the equations in (3) in Subsection 7.2 shows

- $1, y, x, y^2, xy, xy^2$ form a k -basis of $Q_0(\nabla f)$.
- $y^3 = -\frac{3a_{1,3}}{4a_{0,4}}xy^2$
- $x^2 = -\frac{1}{3a_{3,0}}(2a_{2,2}xy^2 + a_{1,3}y^3) = \left(-\frac{2a_{2,2}}{3a_{3,0}} + \frac{a_{1,3}^2}{4a_{3,0}a_{0,4}}\right)xy^2$.
- $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

A direct computation of the Hessian determinant $\det H_f$ in $Q_0(\nabla f)$ gives

$$\det H_f = 72a_{3,0}a_{0,4}xy^2,$$

which is non-zero if $\text{char } k \neq 2, 3$. We then define η by $\eta(xy^2) = a$, with $a = 1/12a_{3,0}a_{0,4}$, and $\eta = 0$ on all other basis elements. Then $\eta(\det H_f) = 6$, the Milnor number.

The Gram matrix is then the upper-triangular anti-diagonal matrix

$$\begin{bmatrix} * & \cdots & * & a \\ \vdots & \ddots & \ddots & 0 \\ * & \ddots & \ddots & \vdots \\ a & 0 & \cdots & 0 \end{bmatrix},$$

with a along the anti-diagonal. By Corollary 2.3, $\mu_0^{\mathbb{A}^1}(f) = 3h$.

Case (iii). We assume that $\Gamma(f)$ has two or three edges. By Lemmas 6.3 and 6.4, we can assume that f has the form

$$(5) \quad \sum_{\substack{i,j \\ 3 \leq i+j \leq 7}} a_{i,j} x^i y^j$$

where

- $a_{1,2} = a_{0,3} = a_{0,4} = 0$,
- $a_{2,1} \neq 0$, and
- $a_{1,3} \neq 0$ or $a_{0,5} \neq 0$.

Since it suffices to consider only Newton nondegenerate cases, we can assume $D \neq 0$, where

$$D = a_{1,3}^2 - 4a_{2,1}a_{0,5}.$$

Row-reduction of these equations (3) (and only using the assumptions following (5) and $D \neq 0$), we obtain

- $1, x, y, y^2, y^3, y^4$ form a k -basis of $Q_0(\nabla f)$.
- $x^2 = \frac{1}{2a_{2,1}^2}(-10a_{2,1}a_{0,5} + 3a_{1,3}^2)y^4$
- xy is a k -linear combination of y^3, y^4
- $xy^2 = -\frac{a_{1,3}}{2a_{2,1}}y^4$
- $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

Using these results, a direct calculation of $\det H_f$ shows that it has only three nonzero terms in $Q_0(\nabla f)$, and they simplify to $\det H_f = -15Dy^4$, which is non-zero if $\text{char } k \neq 3, 5$. We can then define η by $\eta(y^4) = a$, where $a = -2/5D$, and $\eta = 0$ on all other basis elements. Then $\eta(\det H_f) = 6$, the Milnor number.

Using the ordering $\{1, y, y^2, x, y^3, y^4\}$, the corresponding Gram matrix is the block upper anti-triangular matrix:

$$\begin{pmatrix} * & & & & a \\ & & & a & \\ & & B & & \\ & a & & & \\ a & & & & 0 \end{pmatrix}$$

with

$$B = a \begin{bmatrix} 1 & \frac{-a_{1,3}}{2a_{2,1}} \\ \frac{-a_{1,3}}{2a_{2,1}} & \frac{3a_{1,3}^2 - 10a_{0,5}a_{2,1}}{2a_{2,1}^2} \end{bmatrix}.$$

Using Lemma 2.2, $\mu_0^{\mathbb{A}^1}(f) = 2h + q_B$, where q_B is the form corresponding to B . Since the form q_B represents a , $q_B \cong \langle a, \frac{\det B}{a} \rangle$. As $\det B/a$ simplifies to $-\frac{1}{2a_{2,1}^2}$,

$$q_B \cong \langle -\frac{2}{5D}, -\frac{1}{2a_{2,1}^2} \rangle \cong \langle -2, -10D \rangle$$

Thus

$$\mu^{\mathbb{A}^1}(f) \cong 2h + \langle -2, -10D \rangle$$

By choosing the parameters $a_{1,3} = 0$, $a_{2,1} = 1/40$, and $a_{0,5} = c$, we have $-10D = c$. Hence, the symmetric bilinear forms that arise as $\mathbb{A}^1$ -Milnor numbers of rank 6 in case (iii) are exactly those of the form $2h + \langle -2, c \rangle$ for any nonzero c . As this form is $3h$ when $c = 2$, the forms from the previous cases (i) and (ii) appear in this case as well.

7.5. Rank 8. We now assume $\nu(f) = 8$. From Section 6, there are three cases: (i) $\Gamma(f)$ has one edge with vertices $(2, 0), (0, 9)$; (ii) $\Gamma(f)$ has one edge with vertices $(3, 0), (0, 5)$; and (iii) $\Gamma(f)$ has two or three edges.

Case (i). When $m = 2$ and $n = 9$, we can use Lemma 4.2 to delete all terms with total degree at least 10 and consider polynomials of the form

$$f = \sum_{\substack{i,j \\ 2 \leq i+j \leq 9}} a_{i,j} x^i y^j$$

where $a_{1,1} = a_{1,2} = a_{1,3} = a_{1,4} = a_{0,2} = a_{0,3} = a_{0,4} = a_{0,5} = a_{0,6} = a_{0,7} = a_{0,8} = 0$ and $a_{2,0}$ and $a_{0,9}$ are nonzero. The above approach yields that

- $1, y, y^2, y^3, y^4, y^5, y^6, y^7$ form a k -basis of $Q_0(\nabla f)$.
- x is a linear combination of y^5, y^6, y^7
- xy is a linear combination of y^6, y^7
- xy^2 is a multiple of y^7 .
- $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

We now use this to compute the Hessian determinant $\det H_f$ in $Q_0(\nabla f)$. Using the results above, in the direct calculation of $\det H_f$, it is $144a_{2,0}a_{0,9}y^7$, which is non-zero if $\text{char } k \neq 2, 3$. We can then define η by $\eta(y^7) = a$, where $a = 1/(18a_{2,0}a_{0,9})$, and $\eta = 0$ on all other basis elements. Then $\eta(\det H_f) = 8$, the Milnor number.

The Gram matrix is then the antidiagonal matrix

$$\begin{bmatrix} 0 & \cdots & 0 & a \\ \vdots & \ddots & a & 0 \\ 0 & a & \ddots & \vdots \\ a & 0 & \cdots & 0 \end{bmatrix},$$

By Corollary 2.3, this means the $\mathbb{A}^1$ -Milnor number is $4h$.

Case (ii). When $m = 3$ and $n = 5$, we can use Lemma 4.2 and assume

$$f = \sum_{\substack{i,j \\ 3 \leq i+j \leq 9}} a_{i,j} x^i y^j$$

where $a_{0,3} = a_{0,4} = a_{1,2} = a_{1,3} = a_{2,1} = 0$ and $a_{3,0}$ and $a_{0,5}$ are nonzero. The above approach yields that

- $1, y, x, y^2, xy, y^3, xy^2, xy^3$ form a k -basis of $Q_0(\nabla f)$.
- $x^2y = -\frac{2a_{2,2}}{3a_{3,0}}xy^3$
- $y^4 = \frac{1}{5a_{0,5}} \left(\frac{4a_{2,2}^2}{3a_{3,0}} - 4a_{1,4} \right) xy^3$
- $x^2 = -\frac{2a_{2,2}}{3a_{3,0}}xy^2 + \frac{1}{3a_{3,0}} \left(\frac{2a_{3,1}a_{2,2}}{a_{3,0}} - 2a_{2,3} + \frac{a_{1,4}}{5a_{0,5}} \left(-\frac{4a_{2,2}^2}{3a_{3,0}} + 4a_{1,4} \right) \right) xy^3$.
- $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

We now use this to compute the Hessian determinant $\det H_f$ in $Q_0(\nabla f)$. Using these results, a direct calculation of $\det H_f$ gives $\det H_f = 120a_{3,0}a_{0,5}xy^3$, which is non-zero if $\text{char } k \neq 2, 3, 5$. We can then define η by $\eta(xy^3) = a$, where $a = 1/(15a_{3,0}a_{0,5})$, and $\eta = 0$

on all other basis elements. Then $\eta(\det H_f) = 8$, the Milnor number. The Gram matrix is then of the form

$$\begin{bmatrix} * & \cdots & * & a \\ & \ddots & * & a & 0 \\ \vdots & * & \ddots & 0 & 0 \\ * & a & 0 & \ddots & \vdots \\ a & 0 & 0 & \cdots & 0 \end{bmatrix},$$

By Corollary 2.3, this means the $\mathbb{A}^1$ -Milnor number is $4h$.

Case (iii). In this case, we assume that $\mu_0(f) = 8$ and $\Gamma(f)$ has two or three edges. By Lemmas 6.3 and 6.4, we can assume that f has the form

$$(6) \quad f = \sum_{\substack{i,j \\ 3 \leq i+j \leq 9}} a_{i,j} x^i y^j$$

where

- $a_{1,2} = a_{1,3} = a_{0,3} = a_{0,4} = a_{0,5} = a_{0,6} = 0$,
- $a_{2,1} \neq 0$, and
- $a_{1,4} \neq 0$ or $a_{0,7} \neq 0$.

Since it suffices to consider only Newton nondegenerate cases, we can assume $D \neq 0$, where

$$D = a_{1,4}^2 - 4a_{2,1}a_{0,7}.$$

Row reduction of equations in (3) yields

- $1, x, y, y^2, y^3, y^4, y^5, y^6$ form a k -basis of $Q_0(\nabla f)$.
- $x^2 = \frac{1}{a_{2,1}^2}(-7a_{2,1}a_{0,7} + 2a_{1,4}^2)y^6$
- xy is a linear combination of y^4, y^5, y^6
- $xy^2 = -\frac{a_{1,4}}{2a_{2,1}}y^5 + \frac{a_{2,2}a_{1,4} - a_{2,1}a_{1,5}}{2a_{2,1}^2}y^6$
- $xy^3 = -\frac{a_{1,4}}{2a_{2,1}}y^6$
- $\alpha = 0$ in $Q_0(\nabla f)$, for all other $\alpha \in S$.

We now use this to compute the Hessian determinant $\det H_f$ in $Q_0(\nabla f)$. ~~Using the results above,~~ In the direct calculation of $\det H_f$, the only three nonzero terms in $\det H_f$ are:

$$-4a_{2,1}^2x^2 + 8a_{2,1}a_{1,4}xy^3 + (84a_{2,1}a_{0,7} - 16a_{1,4}^2)y^6,$$

which simplifies to $-28Dy^6$, which is nonzero if $\text{char } k \neq 2, 7$. We can then define η by $\eta(y^6) = a$, where $a = -\frac{2}{7D}$, and $\eta = 0$ on all other basis elements. Then $\eta(\det H_f) = 8$, the Milnor number.

Using the ordering $\{1, y, y^2, y^3, x, y^4, y^5, y^6\}$, the Gram matrix is then the block upper anti-triangular matrix:

$$\begin{pmatrix} & & & & a \\ & * & & & a \\ & & & a & \\ & & B & & \\ & a & & & \\ a & & & & 0 \end{pmatrix}$$

with

$$B = a \begin{pmatrix} 1 & -b/2 \\ -b/2 & 2b^2 - 7\frac{a_{0,7}}{a_{2,1}} \end{pmatrix}, \text{ and } b = \frac{a_{1,4}}{a_{2,1}}$$

Using Lemma 2.2, the corresponding symmetric bilinear form is $3h \oplus q_B$, where q_B is the form corresponding to B . Since q_B represents a , $q_B \cong \langle a, \frac{\det B}{a} \rangle$. As $\det B/a$ simplifies to $-\frac{1}{2a_{2,1}^2}$,

$$q_B \cong \left\langle -\frac{2}{7D}, -\frac{1}{2a_{2,1}^2} \right\rangle \cong \langle -2, -14D \rangle$$

Thus

$$\mu^{\mathbb{A}^1}(f) \cong 3h + \langle -2, -14D \rangle$$

By choosing the parameters $a_{1,4} = 0$, $a_{2,1} = 1/56$, and $a_{0,7} = c$, we have $-14D = c$. Hence, the symmetric bilinear forms that arise as $\mathbb{A}^1$ -Milnor numbers of rank 8 in case (iii) are exactly those of the form $3h + \langle -2, c \rangle$ for any nonzero c . When $c = 2$, $\langle -2, c \rangle = h$, and $\mu^{\mathbb{A}^1}(f) = 4h$, recovering the form found in cases (i) and (ii) for rank 8.

Thus the Grothendieck–Witt classes of rank at most 8 that arise as $\mathbb{A}^1$ -Milnor numbers of isolated singularities of plane curves are exactly those given in Table 1.

8. OVER SPECIFIC FIELDS

Thus far, what we have done works over any field of characteristic not 2, 3, 5, or 7. We now explore what happens for specific fields. We have seen in Theorem 3.3 that every EKL form of rank at most 7 arises as an $\mathbb{A}^1$ -Milnor number of a polynomial in three variables $f: \mathbb{A}^3 \rightarrow \mathbb{A}^1$. When $k = \mathbb{C}$, the only form of rank r is $r\langle 1 \rangle$, and we can see from Example 3.1 that every form of rank at most 7 over $\mathbb{C}$ arises as a $\mathbb{A}^1$ -Milnor number from a single-variable function $f: \mathbb{A}^1 \rightarrow \mathbb{A}^1$.

In this section, we show that the situation is different over $\mathbb{R}$ and $\mathbb{Q}$ by proving the following result.

Theorem 8.1. *Let k be $\mathbb{Q}$ or $\mathbb{R}$.*

- (a) *There exist EKL forms of ranks 4, 6, and 8 of morphisms $\mathbb{A}^2 \rightarrow \mathbb{A}^2$ that do not arise as $\mathbb{A}^1$ -Milnor numbers of morphisms $\mathbb{A}^2 \rightarrow \mathbb{A}$.*
- (b) *There exist $\mathbb{A}^1$ -Milnor numbers of ranks 4, 6, and 8 of morphisms $\mathbb{A}^n \rightarrow \mathbb{A}$ that do not arise as $\mathbb{A}^1$ -Milnor numbers of morphisms $\mathbb{A}^2 \rightarrow \mathbb{A}$.*

By [17, Lemma 8], every form $mh + \langle a, b \rangle$ arises as an EKL form of a morphism $\mathbb{A}^2 \rightarrow \mathbb{A}^2$. By Example 3.2, every form $mh + \langle a, b \rangle$ arises as an $\mathbb{A}^1$ -Milnor number of a morphism $\mathbb{A}^3 \rightarrow \mathbb{A}^1$. Thus all forms with signatures (3, 1), (4, 2), or (5, 3) arise in these ways. We will prove Theorem 8.1 by showing in Propositions 8.2, 8.3, and 8.4 that forms with these signatures cannot arise as $\mathbb{A}^1$ -Milnor numbers of morphisms $\mathbb{A}^2 \rightarrow \mathbb{A}^1$.

8.1. Milnor numbers over $\mathbb{R}$. Let $k = \mathbb{R}$. Over $\mathbb{R}$, a symmetric bilinear form q is uniquely determined by its rank and signature.

Proposition 8.2. *Let q be a rank 4 symmetric bilinear form over $\mathbb{R}$. Then q can be an $\mathbb{A}^1$ -Milnor number $\mu_0(f)$ for some $f \in \mathbb{R}[x, y]$ with an isolated singularity at zero if and only if q has signature (2, 2) or (1, 3).*

Proof. ($\Rightarrow$) In Section 7.2, we show (see Table 1) that the $\mathbb{A}^1$ -Milnor numbers of rank 4 are of the form $h + \langle 3\Delta_g A, A \rangle$. We now show that the form $\langle 3\Delta_g A, A \rangle$, where $A = 6a_{3,0}a_{1,2} - 2a_{2,1}^2$ and

$$\Delta_g = -27a_{3,0}^2a_{0,3}^2 + (18a_{3,0}a_{2,1}a_{1,2} - 4a_{2,1}^3)a_{0,3} - 4a_{3,0}a_{1,2}^3 + a_{2,1}^2a_{1,2}^2$$

cannot have signature $(2, 0)$. Suppose it did. Then both A and Δ_g would need to be positive. Now consider Δ_g as a quadratic polynomial in $a_{0,3}$. Its discriminant is

$$-16(3a_{3,0}a_{1,2} - a_{2,1}^2)^3 = -2A^3 < 0,$$

as $A > 0$. Since the leading coefficient of $a_{0,3}^2$ in Δ_g is $-27a_{3,0}^2$, the quadratic Δ_g is negative for all choices of $a_{0,3}$. Hence, it is not possible for A , Δ_g to both be positive. Thus the $\mathbb{A}^1$ -Milnor number cannot have signature $(3, 1)$.

($\Leftarrow$) We have seen in cases (i) and (ii) of Section 7.3, that the form $3h$ with signature $(2, 2)$ can occur. Letting $f(x, y) = x^3 + x^2y - \frac{2}{27}y^3$, then $\mu_0(f)$ has signature $(1, 3)$ (here $A < 0$ and $\Delta_g > 0$). $\square$

Proposition 8.3. *Let q be a rank 6 symmetric bilinear form over $\mathbb{R}$. Then q can be an $\mathbb{A}^1$ -Milnor number $\mu_0(f)$ for some $f \in \mathbb{R}[x, y]$ with an isolated singularity at zero if and only if q has signature $(3, 3)$ or $(2, 4)$.*

Proof. It was shown in Section 7.4 that $\mu_0(f) = 2h + \langle -2, c \rangle$, where c can be chosen to be any non-zero real number. Thus, $\mu_0(f)$ has signature $(3, 3)$ or $(2, 4)$, and both can occur. $\square$

Proposition 8.4. *Let q be a rank 8 symmetric bilinear form over $\mathbb{R}$. Then q can be a $\mathbb{A}^1$ -Milnor number $\mu_0(f)$ for some $f \in \mathbb{R}[x, y]$ with an isolated singularity at zero if and only if q has signature $(4, 4)$ or $(3, 5)$.*

Proof. It was shown in Section 7.5 that $\mu_0(f) = 3h + \langle -2, c \rangle$, where c can be chosen to be any non-zero real number. Thus, $\mu_0(f)$ has signature $(4, 4)$ or $(3, 5)$, and both can occur. $\square$

Proof of Theorem 8.1. By Propositions 8.2, 8.3, and 8.4, forms with signature $(3, 1)$, $(4, 2)$, or $(5, 3)$ cannot arise as $\mathbb{A}^1$ -Milnor numbers. As [17, Lemma 8] shows that there are EKL forms with these signatures, the theorem follows. $\square$

For the rest of this section, we work over $\mathbb{Q}$ and for right now only consider rank 6. In this case, we know that the $\mathbb{A}^1$ -Milnor number has the form $2h + \langle -2, c \rangle$, and any $c \in \mathbb{Q}$ is possible. We now classify which symmetric bilinear forms over $\mathbb{Q}$ are of the form $\langle -2, c \rangle$.

8.2. $\mathbb{A}^1$ -Milnor numbers of rank 6 and 8 over $\mathbb{Q}$ coming from bivariate polynomials. Let $f \in \mathbb{Q}[x, y]$ have an isolated singularity at zero with $\mu_0(f) = 6$. In Section 7.4, we showed $\mu_0(f) = 2h + \langle -2, c \rangle$, where $c \in \mathbb{Q}$ can be chosen to be any non-zero number. We now determine the symmetric bilinear forms over $\mathbb{Q}$ of rank 6 that are $\mathbb{A}^1$ -Milnor numbers by determining which rank 2 symmetric bilinear forms are equivalent to $\langle -2, c \rangle$.

We refer to Serre [19] for the background on symmetric bilinear forms over $\mathbb{Q}$ and the p -adic fields $\mathbb{Q}_p$ (note that Serre's treatment is in terms of quadratic forms, which are equivalent to symmetric bilinear forms over k when $\text{char } k \neq 2$). We recall that a symmetric bilinear form q of rank 2 over $\mathbb{Q}$ is determined by its discriminant $d_q \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$, its signature $(2 - s, s)$, and the Hasse-Witt invariants $\varepsilon_p = \pm 1$, for each prime p , including $p = \infty$.

- (1) $d_q \equiv (-1)^s$ in $\mathbb{R}/(\mathbb{R})^2$,
- (2) $\varepsilon_\infty = (-1)^{s(s-2)/2}$,
- (3) If $d_q \equiv -1$ in $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, then $\varepsilon_p = 1$.

- (4) $\varepsilon_p = -1$, for only a finite number of p
- (5) $\prod_p \varepsilon_p = 1$.

Conversely, if these conditions are met, there is a symmetric bilinear form of rank 2 over $\mathbb{Q}$ with the invariants $\{d_q, (2-s, s), \varepsilon_p\}$.

We now characterize the symmetric bilinear forms of the form $\langle a, b \rangle$ that are equivalent to $\langle -2, c \rangle$. For a prime p and a non-zero integer n , we define $v_p(n)$ to be the exact power of the prime p dividing n . We extend the domain of v_p to rational numbers a/b by defining $v_p(a/b) = v_p(a) - v_p(b)$.

Theorem 8.5. *The symmetric bilinear forms $q = \langle -2, c \rangle$, for $c \in \mathbb{Q}^*$ are the rank 2 symmetric bilinear forms determined by the invariants $(d_q, (2-s, s), \varepsilon_p)$ satisfying*

- (1) $s = 1$ or 2
- (2) $d_q \equiv (-1)^s$ and $\varepsilon_\infty \equiv -d_q$ in $\mathbb{R}^*/(\mathbb{R}^*)^2$,
- (3) If p is an odd prime and $p \equiv 5, 7 \pmod{8}$, then $\varepsilon_p = (-1)^{v_p(d_q)}$.
- (4) If p is an odd prime and $p \equiv 1, 3 \pmod{8}$, then $\varepsilon_p = 1$.
- (5) $\varepsilon_2 = \prod_{p \neq 2} \varepsilon_p$ (including $p = \infty$)

Proof. ($\Rightarrow$) Assuming $q = \langle -2, c \rangle$, we will prove that the conditions (1) through (5) are satisfied. First, $d_q = -2c$. If $c > 0$, then the signature of q is $(1, 1)$, $s = 1$, and $d_q \equiv (-1)^s$ in $\mathbb{R}^*/(\mathbb{R}^*)^2$. Also $\varepsilon_\infty = (-2, c)_\infty = 1 \equiv -d_q$. If $c < 0$, $s = 2$ and $d_q \equiv 1 = (-1)^s$. Also, $\varepsilon_\infty = (-2, c) = -1 \equiv -d$. Hence (1) and (2) hold true. Also (5) is true for any form q . Now let p be an odd prime let $\left(\frac{-2}{p}\right)$ be the Legendre symbol. By Quadratic reciprocity, it equals 1, if $p \equiv 1, 3 \pmod{8}$ and is -1 if $p \equiv 5, 7 \pmod{8}$. By [19, Chap. III, Thm 1],

$$\varepsilon_p = \left(\frac{-2}{p}\right)^{v_p(c)} = \left(\frac{-2}{p}\right)^{v_p(d_q)},$$

proving statements (3) and (4).

($\Leftarrow$) Conversely, we now assume that conditions (1)-(5) are satisfied. For clarity, let d, s', ε'_p denote a set of values satisfying these conditions. Let $c = -2d$ and define $q = \langle -2, c \rangle$. We will show that $d_q \equiv d$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, $s_q = s'$, and $\varepsilon_{q,p} = \varepsilon'_p$ for all p .

We note that $d_q = 4d \equiv d$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$. If $d > 0$, then $s' = 2$. Now q has signature $(0, 2)$ with $s_q = s'$. Since $d_q = 4d > 0$, and $d_q \equiv (-1)^{s_q}$. And $\varepsilon_{q,\infty} = (-2, -2d_q) = -1 \equiv -d_q \equiv -d \equiv \varepsilon'_\infty$. When $d_q < 0$, we have the same results with $s = 1$.

For an odd prime p , since $v_p(d_q) = v_p(4d) = v_p(d)$, the same argument as above shows that $\varepsilon_{q,p} = \varepsilon'_p$. Finally, as

$$\varepsilon_{q,2} = \prod_{p \neq 2} \varepsilon_{q,p} = \prod_{p \neq 2} \varepsilon'_p = \varepsilon'_2$$

we see that the invariants of the form q have the specified properties. $\square$

As the invariants of the forms $nh + \langle -2, c \rangle$ are determined by those of $\langle -2, c \rangle$, we obtain:

Corollary 8.6. *The symmetric bilinear forms $q = 2h + \langle -2, c \rangle$, for $c \in \mathbb{Q}^*$ are the rank 6 symmetric bilinear forms with the invariants $(d_q, (2-s, s), \varepsilon_p)$ satisfying*

- (1) $s = 3$ or 4
- (2) $\varepsilon_\infty \equiv d_q \equiv (-1)^s$ in $\mathbb{R}^*/(\mathbb{R}^*)^2$,
- (3) If p is an odd prime, $p \equiv 5, 7 \pmod{8}$, then $\varepsilon_p = (-1)^{v_p(d_q)}$.
- (4) If p is an odd prime, $p \equiv 1, 3 \pmod{8}$, then $\varepsilon_p = 1$.
- (5) $\varepsilon_2 = \prod_{p \neq 2} \varepsilon_p$ (including $p = \infty$)

Corollary 8.7. *The symmetric bilinear forms $q = 3h + \langle -2, c \rangle$, for $c \in \mathbb{Q}^*$ are the rank 8 symmetric bilinear forms with the invariants $(d_q, (2 - s, s), \varepsilon_p)$ satisfying*

- (1) $s = 4$ or 5 and $\varepsilon_\infty = 1$;
- (2) $d_q \equiv (-1)^s$ in $\mathbb{R}^*/(\mathbb{R}^*)^2$,
- (3) If p is an odd prime, $p \equiv 5, 7 \pmod{8}$, then $\varepsilon_p = (-1)^{v_p(d_q)}$.
- (4) If p is an odd prime, $p \equiv 1, 3 \pmod{8}$, then $\varepsilon_p = 1$.
- (5) $\varepsilon_2 = \prod_{p \neq 2} \varepsilon_p$ (including $p = \infty$)

9. A DIMENSION REDUCTION TECHNIQUE

In this section, we present a reduction technique for $\mathbb{A}^1$ -Milnor numbers of the type that has been used for local $\mathbb{A}^1$ -Brouwer degrees. First, we recall the following result of this type from Quick–Strand–Wilson [17].

Theorem 9.1 ([17, Theorem 15]). *Suppose $F: \mathbb{A}^n \rightarrow \mathbb{A}^n$ is a map with an isolated zero at 0 and suppose that the rank of $Q_0(F)$ is $N \geq 1$. If the ideal $(F_1, \dots, F_n)$ is not contained in $(x_1, \dots, x_n) \subseteq k[x_1, \dots, x_n]$, then we can eliminate a variable in the description of $Q_0(F)$. That is, there is a map $G: \mathbb{A}^{n-1} \rightarrow \mathbb{A}^{n-1}$ with isolated zero at 0 and $Q_0(G) \cong Q_0(F)$. Furthermore, the EKL form of F and G differ only by multiplication by a unit.*

In [17], Quick, Strand, and Wilson first found all possible shapes of the EKL forms of rank up to seven in two variables. Then they applied the above Theorem 9.1 to conclude that increasing the number of variables does not produce any new forms. As they point out in [17, Remark 16], the method of such reduction was introduced by McKean [13, Lemma 5.7] when calculating the local $\mathbb{A}^1$ -Brouwer degree of the intersection of two plane curves. In both [13] and [17], the reduction shows the existence of a function with fewer variables and an isomorphic local ring, such that the $\mathbb{A}^1$ -Brouwer degree is equal up to multiplication by a unit. By reducing the number of variables needed, these theorems can often help to reduce the complexity of computations.

When the problem is to compute an $\mathbb{A}^1$ -Milnor number instead of an EKL form, one lets $F = \nabla f$, where $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$. Now the F_j are the partial derivatives f_{x_j} , which are no longer independent of each other, as in the EKL situation. As a result, the situation becomes much more subtle. Nevertheless, we are able to prove in Theorem 9.6 a reduction theorem for $\mathbb{A}^1$ -Milnor numbers.

Definition 9.2. Let R be a local ring with maximal ideal $\mathfrak{m}$. If $f \in \mathfrak{m}$, the *order* of f is defined to be the largest integer k such that $f \in \mathfrak{m}^k$. By convention, the zero polynomial has order ∞ .

On the way to proving Theorem 9.6, we begin with a lemma that shows that one can apply a change of variables without changing the $\mathbb{A}^1$ -Milnor number. The main purpose of this lemma is to transform f such that it does not have the term x_n in degree 1. Applying this lemma once will increase the total degree of the coefficient of x_n . By repeating this procedure, the degree gets large enough and vanishes in $Q_0(\nabla f)$.

Lemma 9.3. *Let $n, s \geq 2$ be integers, let a be a nonzero element of k , and let $\alpha_i \in k[x_1, \dots, x_{n-1}]$ be polynomials for $i \in \{0, 1, \dots, s\}$. If $\alpha_1 \neq 0$ and $\alpha_1(0) = \alpha_2(0) = 0$, then there exist polynomials $\beta_i \in k[x_1, \dots, x_{n-1}]$ for $i \in \{0, 1, \dots, s\}$ with $\beta_1(0) = \beta_2(0) = 0$,*

$$Q_0 \left(\nabla \left(ax_n^2 + \sum_{i=0}^s \alpha_i \cdot x_n^i \right) \right) = Q_0 \left(\nabla \left(ax_n^2 + \sum_{i=0}^s \beta_i \cdot x_n^i \right) \right),$$

$$\mu_0^{\mathbb{A}^1} \left(ax_n^2 + \sum_{i=0}^s \alpha_i \cdot x_n^i \right) = \mu_0^{\mathbb{A}^1} \left(ax_n^2 + \sum_{i=0}^s \beta_i \cdot x_n^i \right),$$

and $\text{ord}(\beta_1) > \text{ord}(\alpha_1)$.

Proof. By completing the square, letting $x_n = X_n - \frac{\alpha_1}{2a}$, and expanding, we have

$$\begin{aligned} ax_n^2 + \sum_{i=0}^s \alpha_i \cdot x_n^i &= a \left(x_n + \frac{\alpha_1}{2a} \right)^2 - \frac{\alpha_1^2}{4a^2} + \alpha_0 + \sum_{i=2}^s \alpha_i \cdot x_n^i \\ &= aX_n^2 + \alpha_0 - \frac{\alpha_1^2}{4a^2} + \sum_{i=2}^s \alpha_i \cdot \left(X_n - \frac{1}{2}\alpha_1 \right)^i \\ &= aX_n^2 + \alpha_0 - \frac{\alpha_1^2}{4a^2} + \sum_{i=2}^s \alpha_i \cdot \left(-\frac{1}{2}\alpha_1 \right)^i + \sum_{i=2}^s i\alpha_i \cdot \left(-\frac{1}{2}\alpha_1 \right)^{i-1} X_n \\ &\quad + \sum_{j=2}^s \sum_{i=j}^s \binom{i}{j} \alpha_i \cdot \left(-\frac{1}{2}\alpha_1 \right)^{i-j} X_n^j. \end{aligned}$$

This is of the form

$$aX_n^2 + \sum_{j=0}^s \beta_j \cdot X_n^j$$

with

$$\beta_j = \begin{cases} \alpha_0 - \frac{\alpha_1^2}{4a^2} + \sum_{i=2}^s \alpha_i \cdot \left(-\frac{1}{2}\alpha_1 \right)^i & \text{for } j = 0, \\ \sum_{i=2}^s i\alpha_i \cdot \left(-\frac{1}{2}\alpha_1 \right)^{i-1} & \text{for } j = 1, \\ \sum_{i=j}^s \binom{i}{j} \alpha_i \cdot \left(-\frac{1}{2}\alpha_1 \right)^{i-j} & \text{for } j \geq 2. \end{cases}$$

Note that $\alpha_1(0) = 0$ implies $\beta_1(0) = 0$ and that $\alpha_1(0) = \alpha_2(0) = 0$ implies $\beta_2(0) = 0$. Theorem 5.3 gives that

$$Q_0 \left(\nabla \left(ax_n^2 + \sum_{i=0}^s \alpha_i \cdot x_n^i \right) \right) = Q_0 \left(\nabla \left(X_n^2 + \sum_{j=0}^r \beta_j \cdot X_n^j \right) \right)$$

and

$$\mu_0^{\mathbb{A}^1} \left(ax_n^2 + \sum_{i=0}^r \alpha_i \cdot x_n^i \right) = \mu_0^{\mathbb{A}^1} \left(aX_n^2 + \sum_{j=0}^r \beta_j \cdot X_n^j \right).$$

We now show that $\text{ord}(\beta_1) > \text{ord}(\alpha_1)$. By definition, β_1 is the sum of terms $\alpha_i \alpha_2^{i-1}$, with $i \geq 2$, and for each i , we will show $\text{ord}(\alpha_i \alpha_2^{i-1}) > \text{ord}(\alpha)$. By the hypothesis on α_1 , we have $1 \leq \text{ord}(\alpha_1) < \infty$. Since $\text{ord}(\alpha_2) \geq 1$, we have

$$\text{ord}(\alpha_2 \cdot \alpha_1) = \text{ord}(\alpha_2) + \text{ord}(\alpha_1) > \text{ord}(\alpha_1).$$

For $i \geq 3$, we have $\text{ord}(\alpha_i \cdot \alpha_1^{i-1}) \geq \text{ord}(\alpha_1^{i-1}) > \text{ord}(\alpha_1)$. Thus $\text{ord}(\beta_1) > \text{ord}(\alpha_1)$, completing the proof. $\square$

We illustrate the above with an example.

Example 9.4. Consider $f = (y + x^2)^2 + x^3y + x^3y^2$. This is of the form in Lemma 9.3 with $a = 1$, $\alpha_0 = x^4$, $\alpha_1 = 2x^2 + x^3$, $\alpha_2 = x^3$, and $\alpha_i = 0$ for all $i \geq 3$. Therefore we can apply Lemma 9.3 to get another function with the same $\mathbb{A}^1$ -Milnor number, where $\beta_0 = x^4 - \frac{(2x^2+x^3)^2}{4} + \frac{(2x^2+x^3)^2}{4}x^3$, $\beta_1 = -(2x^2 + x^3)x^3$, $\beta_2 = x^3$, and $\beta_i = 0$ for all $i \geq 3$. In this particular example, note that the order of β_1 is 5, strictly bigger than that of α_1 . Moreover, the order β_1y is 6. Since $\dim Q_0 = 4$, $\dim Q_0 + 2 = 6$ and by Lemma 4.2, the term can be deleted without changing the $\mathbb{A}^1$ -Milnor number.

In general by repeatedly applying Lemma 9.3, we can make the order of α_1 sufficiently large that the term $\alpha_1 \cdot x_n$ can be disregarded by Lemma 4.2, so we can assume $\alpha_1 = 0$ as it will be done in the next Lemma 9.5.

Lemma 9.5. *Let $f: \mathbb{A}^n \rightarrow \mathbb{A}$ be a morphism with isolated singularity at the origin and let $\mathfrak{m} = (x_1, \dots, x_n)$. Assume that $(f_{x_1}, \dots, f_{x_n}) \not\subseteq \mathfrak{m}^2$. Then there exists a map $h: \mathbb{A}^n \rightarrow \mathbb{A}$ with isolated singularity at the origin of the form*

$$h = \alpha_0 + ax_n^2 + \sum_{i=2}^s \alpha_i \cdot x_n^i,$$

for a nonzero scalar a and $\alpha_i \in k[x_1, \dots, x_{n-1}]$ with $\text{ord}(\alpha_2) \geq 1$ such that $Q_0(\nabla f) \cong Q_0(\nabla h)$ and

$$\mu_0^{\mathbb{A}^1}(f) = \mu_0^{\mathbb{A}^1}(h).$$

Proof. Since f has an isolated singularity at the origin, the partial derivatives $f_{x_1}, \dots, f_{x_n}$ have no constant terms. Since $(f_{x_1}, \dots, f_{x_n}) \not\subseteq \mathfrak{m}^2$, one of $f_{x_1}, \dots, f_{x_n}$ has a nonzero linear term and thus f contains at least one nonzero monomial with degree 2. These terms are of the form ax_i^2 or ax_ix_j . If f does not contain a non-zero monomial ax_i^2 for some i , then it must have a nonzero term ax_ix_j for some $i \neq j$. In this case, by using a change of basis and replacing x_j by $x_i + x_j$, we have a polynomial f' with a nonzero monomial ax_i^2 . By Theorem 5.3, $\mu_0^{\mathbb{A}^1}(f) = \mu_0^{\mathbb{A}^1}(f')$. By replacing f by f' , we can assume that f contains the non-zero term ax_i^2 for some i and $a \neq 0$. By renumbering the variables, we can assume f contains the term ax_n^2 , with $a \neq 0$. Thus, we may assume

$$(7) \quad f = \alpha_0 + \alpha_1 \cdot x_n + ax_n^2 + \sum_{i=2}^s \alpha_i \cdot x_n^i$$

for some polynomials $\alpha_i \in k[x_1, \dots, x_{n-1}]$, with $\alpha_1(0) = \alpha_2(0) = 0$, and some $s \geq 2$. If $n = 1$, then α_1 is a constant and $\alpha_1 = 0$, and we let $h = f$ to prove the lemma. The same holds if $n > 1$ and $\alpha_1 = 0$. If $n > 1$ and $\alpha_1 \neq 0$, the polynomial f satisfies the hypotheses of Lemma 9.3. By repeatedly applying Lemma 9.3, we can find a polynomial $h \in k[x_1, \dots, x_n]$ such that

$$(8) \quad h = \beta_0 + \beta_1 \cdot x_n + ax_n^2 + \sum_{i=2}^s \beta_i \cdot x_n^i,$$

with $\beta_i \in k[x_1, \dots, x_{n-1}]$, $\text{ord}(\beta_1) \geq \mu_0(f) + 1$, $\beta_2(0) = 0$, and $\mu_0^{\mathbb{A}^1}(f) = \mu_0^{\mathbb{A}^1}(h)$. Then as $\text{ord}(\beta_1 x_n) \geq \mu_0(h) + 2$, Lemma 4.2 allows us to delete the $\beta_1 x_n$ term from h without changing its $\mathbb{A}^1$ -Milnor number. Thus, we can assume that $h = \beta_0 + ax_n^2 + \sum_{i=2}^s \beta_i \cdot x_n^i$, with $\text{ord}(\beta_2) \geq 1$, which proves the lemma. $\square$

Theorem 9.6. *Let $f: \mathbb{A}^n \rightarrow \mathbb{A}$ be a morphism with isolated singularity at the origin and let $\mathfrak{m} = (x_1, \dots, x_n)$. Assume that $(f_{x_1}, \dots, f_{x_n}) \not\subseteq \mathfrak{m}^2$. Then there exists a map $g: \mathbb{A}^{n-1} \rightarrow \mathbb{A}$ with isolated singularity at the origin and a unit ε such that $Q_0(\nabla f) \cong Q_0(\nabla g)$ and*

$$\mu_0^{\mathbb{A}^1}(f) = \langle \varepsilon \rangle \cdot \mu_0^{\mathbb{A}^1}(g).$$

Proof. By Lemma 9.5, we can assume that f has the form

$$\beta_0 + ax_n^2 + \sum_{i=2}^s \beta_i \cdot x_n^i,$$

with $a \neq 0$, $\beta_i \in k[x_1, \dots, x_{n-1}]$ and $\text{ord}(\beta_2) \geq 1$. Let $g = \beta_0$. Differentiating, we have

$$\begin{aligned} f_{x_j} &= g_{x_j} + \sum_{i=2}^r \frac{\partial \beta_i}{\partial x_j} x_n^i \text{ for } j \in \{1, \dots, n-1\} \\ f_{x_n} &= 2ax_n + \sum_{i=2}^r i\beta_i \cdot x_n^{i-1} = x_n \left(2a + \sum_{i=2}^r i\beta_i \cdot x_n^{i-2} \right). \end{aligned}$$

As ideals in the local ring $k[x_1, \dots, x_n]_{\mathfrak{m}}$, we have

$$\begin{aligned} (f_{x_1}, \dots, f_{x_n}) &= \left(f_{x_1}, \dots, f_{x_{n-1}}, x_n \left(2a + \sum_{i=2}^r i\beta_i \cdot x_n^{i-2} \right) \right) \\ &= (f_{x_1}, \dots, f_{x_{n-1}}, x_n) \\ &= (g_{x_1}, \dots, g_{x_{n-1}}, x_n), \end{aligned}$$

where the second equality holds because $2a + \sum_{i=2}^r i\beta_i \cdot x_n^{i-2}$ is a unit in $k[x_1, \dots, x_n]_{\mathfrak{m}}$. The last equality holds because f_{x_i} is generated by g_{x_i} and x_n and so is g_{x_i} by f_{x_i} and x_n . We then have

$$\begin{aligned} Q_0(\nabla f) &= k[x_1, \dots, x_n] / (f_{x_1}, \dots, f_{x_n})_{\mathfrak{m}} \\ &= k[x_1, \dots, x_n] / (g_{x_1}, \dots, g_{x_{n-1}}, x_n)_{\mathfrak{m}} \\ &= k[x_1, \dots, x_{n-1}] / (g_{x_1}, \dots, g_{x_{n-1}})_{\mathfrak{m}'} \\ &= Q_0(\nabla g), \end{aligned}$$

where $\mathfrak{m}'$ is the ideal $(x_1, \dots, x_{n-1})$ in $k[x_1, \dots, x_{n-1}]$, and we identify elements in both $Q_0(\nabla f)$ and $Q_0(\nabla g)$ as equal. If $\det H_f$ is the Hessian matrix of f and H_g is the Hessian matrix of g , we have that

$$H_f = \left(\begin{array}{c|c} H_g & 0 \\ \hline 0 & 2a \end{array} \right),$$

where the zeros occur because x_n is 0 in $Q_0(\nabla f)$. Therefore we have $\det H_f = 2a \det(H_g)$ under the isomorphism.

We define η_g so that $\eta_g(\det H_g) = \mu_0(g)$ and define $\eta_f(y) = \frac{1}{2a} \eta_g(y)$ for any $y \in Q_0(\nabla f)$. Then

$$\begin{aligned} \eta_f(\det H_f) &= \frac{1}{2a} \eta_g(\det H_f) = \frac{1}{2a} \eta_g(2a \det H_g) \\ &= \eta_g(\det H_g) = \mu_0(g) = \mu_0(f). \end{aligned}$$

Now we can compute the $\mathbb{A}^1$ -Milnor number of f and g using η_f and η_g respectively on the same set of basis elements. Each entry in the Gram matrix of $\mu_0^{\mathbb{A}^1}(f)$ is exactly $\frac{1}{2a}$ the entry of $\mu_0^{\mathbb{A}^1}(g)$ at the same position. Thus $\mu_0^{\mathbb{A}^1}(f) = \langle \frac{1}{2a} \rangle \cdot \mu_0^{\mathbb{A}^1}(g)$. As a^n is a unit, the theorem is proved. $\square$

Example 9.7. Continuing Example 9.4, where $f = (y + x^2)^2 + x^3y + x^3y^2$, Theorem 9.6 produces

$$g = x^4 - \frac{(2x^2 + x^3)^2}{4} + \frac{(2x^2 + x^3)^2}{4}x^3.$$

Since $a = 1$ here, Theorem 9.6 gives that $\mu_0^{\mathbb{A}^1}(f) = \langle \frac{1}{2} \rangle \cdot \mu_0^{\mathbb{A}^1}(g)$.

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