Growth of Integer Sequences

A Perspective Based on Multiplication

Michael Brilleslyper Bradley Warner A. J. Wallerstein

Department of Mathematical Sciences
United States Air Force Academy

Mathfest, 2012

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \geq 6 \cdot 7$, but $\ldots$

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \geq 6 \cdot 7$, but ...
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 < 6 \cdot 7 \cdot 8$

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \geq 6 \cdot 7$, but ...
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 < 6 \cdot 7 \cdot 8$

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \geq 6 \cdot 7$, but ...
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 < 6 \cdot 7 \cdot 8$

We are interested in the maximum number of terms on the right hand side that maintains the inequality for any number of terms on the left side.

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \geq 6 \cdot 7$, but ...
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 < 6 \cdot 7 \cdot 8$

We are interested in the maximum number of terms on the right hand side that maintains the inequality for any number of terms on the left side.

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \geq 7 \cdot 8 \cdot 9$ $\qquad n = 6,\ f(6) = 3$

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \geq 6 \cdot 7$, but ...
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 < 6 \cdot 7 \cdot 8$

We are interested in the maximum number of terms on the right hand side that maintains the inequality for any number of terms on the left side.

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \geq 7 \cdot 8 \cdot 9$ $\quad n = 6,\ f(6) = 3$
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \geq 8 \cdot 9 \cdot 10$ $\quad n = 7,\ f(7) = 3$

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \geq 6 \cdot 7$, but ...
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 < 6 \cdot 7 \cdot 8$

We are interested in the maximum number of terms on the right hand side that maintains the inequality for any number of terms on the left side.

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \geq 7 \cdot 8 \cdot 9$ $\quad n = 6,\ f(6) = 3$
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \geq 8 \cdot 9 \cdot 10$ $\quad n = 7,\ f(7) = 3$
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \geq 9 \cdot 10 \cdot 11 \cdot 12$ $\quad n = 8,\ f(8) = 4$

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \geq 6 \cdot 7$, but ...
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 < 6 \cdot 7 \cdot 8$

We are interested in the maximum number of terms on the right hand side that maintains the inequality for any number of terms on the left side.

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \geq 7 \cdot 8 \cdot 9$ $\quad n = 6,\ f(6) = 3$
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \geq 8 \cdot 9 \cdot 10$ $\quad n = 7,\ f(7) = 3$
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \geq 9 \cdot 10 \cdot 11 \cdot 12$ $\quad n = 8,\ f(8) = 4$

Motivation

Consider the sequence of natural numbers: $1, 2, 3, 4, 5, 6, \ldots$

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \geq 6 \cdot 7$, but ...
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 < 6 \cdot 7 \cdot 8$

We are interested in the maximum number of terms on the right hand side that maintains the inequality for any number of terms on the left side.

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \geq 7 \cdot 8 \cdot 9$ $\quad n = 6,\ f(6) = 3$
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \geq 8 \cdot 9 \cdot 10$ $\quad n = 7,\ f(7) = 3$
- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \geq 9 \cdot 10 \cdot 11 \cdot 12$ $\quad n = 8,\ f(8) = 4$

We call f the *growth function* associated to a particular sequence.

Growth Function Table for the Natural Numbers

n	$f(n)$	n	$f(n)$
20	12	29	19
21	13	30	20
22	14	31	20
23	15	32	21
24	15	33	22
25	16	34	23
26	17	35	23
27	17	36	24
28	18	37	25

Growth Function Table for the Natural Numbers

n	$f(n)$	n	$f(n)$
20	12	29	19
21	13	30	20
22	14	31	20
23	15	32	21
24	15	33	22
25	16	34	23
26	17	35	23
27	17	36	24
28	18	37	25

$37! = 13763753091226345046315979581580902400000000 \geq \prod_{k=38}^{62} k = 2286438375623605083865999463264944128000000$

Definitions

- Let $\{a_k\}$ be a strictly increasing sequence of positive integers. Define the pseudo-factorial function:

$$P^{(n)} = \prod_{k=1}^{n} a_k$$

Definitions

- Let $\{a_k\}$ be a strictly increasing sequence of positive integers. Define the pseudo-factorial function:

$$P^{(n)} = \prod_{k=1}^{n} a_k$$

- For every positive integer n, define $f(n)$ to be the largest integer such that

$$P^{(n)} \geq Q^{(n)} = \prod_{k=n+1}^{n+f(n)} a_k$$

Definitions

- Let $\{a_k\}$ be a strictly increasing sequence of positive integers. Define the pseudo-factorial function:

$$P^{(n)} = \prod_{k=1}^{n} a_k$$

- For every positive integer n, define $f(n)$ to be the largest integer such that

$$P^{(n)} \geq Q^{(n)} = \prod_{k=n+1}^{n+f(n)} a_k$$

- Thus $f(n)$ is the number of terms in the product $Q^{(n)}$

Definitions

- Let $\{a_k\}$ be a strictly increasing sequence of positive integers. Define the pseudo-factorial function:

$$P^{(n)} = \prod_{k=1}^{n} a_k$$

- For every positive integer n, define $f(n)$ to be the largest integer such that

$$P^{(n)} \geq Q^{(n)} = \prod_{k=n+1}^{n+f(n)} a_k$$

- Thus $f(n)$ is the number of terms in the product $Q^{(n)}$
- If no such integer exists, then define $f(n) = 0$.

Exponential Sequence

Consider the exponential sequence consisting of powers of two:

$$1, 2, 4, 8, 16, 32, 64, \ldots$$

Exponential Sequence

Consider the exponential sequence consisting of powers of two:

$$1, 2, 4, 8, 16, 32, 64, \ldots$$

n	$f(n)$	n	$f(n)$
1	0	10	4
2	0	11	4
3	1	12	5
4	1	13	5
5	2	14	6
6	2	15	6
7	3	16	6
8	3	17	7
9	3	18	7

Properties of the Growth Function

Given a strictly increasing sequence of positive integers $\{a_k\}$, what is the general relationship between $f(n)$ and $f(n+1)$?

Properties of the Growth Function

Given a strictly increasing sequence of positive integers $\{a_k\}$, what is the general relationship between $f(n)$ and $f(n+1)$?

Theorem: For all n, $f(n+1) = f(n)$ or $f(n+1) = f(n) \pm 1$

Properties of the Growth Function

Given a strictly increasing sequence of positive integers $\{a_k\}$, what is the general relationship between $f(n)$ and $f(n+1)$?

Theorem: For all n, $f(n+1) = f(n)$ or $f(n+1) = f(n) \pm 1$
Proof: See our paper.

Properties of the Growth Function

Given a strictly increasing sequence of positive integers $\{a_k\}$, what is the general relationship between $f(n)$ and $f(n+1)$?

Theorem: For all n, $f(n+1) = f(n)$ or $f(n+1) = f(n) \pm 1$
Proof: See our paper.

Theorem: For the sequence of natural numbers: $1, 2, 3, 4, 5, \ldots$

- $f(n+1) = f(n)$ or $f(n+1) = f(n) + 1$.

Properties of the Growth Function

Given a strictly increasing sequence of positive integers $\{a_k\}$, what is the general relationship between $f(n)$ and $f(n+1)$?

Theorem: For all n, $f(n+1) = f(n)$ or $f(n+1) = f(n) \pm 1$
Proof: See our paper.

Theorem: For the sequence of natural numbers: $1, 2, 3, 4, 5, \ldots$

- $f(n+1) = f(n)$ or $f(n+1) = f(n) + 1$.
- There exist arbitrarily long (but finite) consecutive runs of integers such that $f(n+1) = f(n) + 1$.

Properties of the Growth Function

Given a strictly increasing sequence of positive integers $\{a_k\}$, what is the general relationship between $f(n)$ and $f(n+1)$?

Theorem: For all n, $f(n+1) = f(n)$ or $f(n+1) = f(n) \pm 1$
Proof: See our paper.

Theorem: For the sequence of natural numbers: $1, 2, 3, 4, 5, \ldots$

- $f(n+1) = f(n)$ or $f(n+1) = f(n) + 1$.
- There exist arbitrarily long (but finite) consecutive runs of integers such that $f(n+1) = f(n) + 1$.
- $f(n)$ remains constant for no more than two steps.

Measuring the Growth of Integer Sequences

Given an increasing sequence of positive integers, we define the *growth limit*, L, to be the quantity

$$L = \lim_{n \to \infty} \frac{f(n)}{n}$$

Given an increasing sequence of positive integers, we define the *growth limit*, L, to be the quantity

$$L = \lim_{n\to\infty} \frac{f(n)}{n}$$

Results:

- Theorem: For the family of sequences $a_k = k^p$, we have $L = 1$.

Measuring the Growth of Integer Sequences

Given an increasing sequence of positive integers, we define the *growth limit*, L, to be the quantity

$$L = \lim_{n \to \infty} \frac{f(n)}{n}$$

Results:

- Theorem: For the family of sequences $a_k = k^p$, we have $L = 1$.
- There exist sequences for which $L = 0$ (Sylvester's Sequence or any doubly exponential sequence). $1, 2, 3, 7, 43, \ldots$

Given an increasing sequence of positive integers, we define the *growth limit*, L, to be the quantity

$$L = \lim_{n \to \infty} \frac{f(n)}{n}$$

Results:

- Theorem: For the family of sequences $a_k = k^p$, we have $L = 1$.
- There exist sequences for which $L = 0$ (Sylvester's Sequence or any doubly exponential sequence). $1, 2, 3, 7, 43, \ldots$
- Conjecture: The sequence of primes appear to have $L = 1$.

Given an increasing sequence of positive integers, we define the *growth limit*, L, to be the quantity

$$L = \lim_{n \to \infty} \frac{f(n)}{n}$$

Results:

- Theorem: For the family of sequences $a_k = k^p$, we have $L = 1$.
- There exist sequences for which $L = 0$ (Sylvester's Sequence or any doubly exponential sequence). $1, 2, 3, 7, 43, \ldots$
- Conjecture: The sequence of primes appear to have $L = 1$.
- Sequences of the form $a_k = 2^{k^p}$ have a discrete set of L-values that tend to zero as $p \to \infty$.

The Limit of $f(n)/n$.

The table below shows some exact and approximate values for the limit of $f(n)/n$ as $n \to \infty$. We observe that there appears to be an infinite, but (possibly) discrete set of values:

a_k	$L = \lim_{n\to\infty} f(n)/n$
k	1
k^p	1
2^k	0.4142
$k!$	0.3755
k^k	0.3755
2^{k^2}	0.2590
2^{k^3}	0.1877
$\vdots$	$\vdots$
2^{2^k}	0

Conjectures, Open Problems, Further Investigations

Given: $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n) < n$ and $|f(n+1) - f(n)| \leq 1$.

Given: $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n) < n$ and $|f(n+1) - f(n)| \leq 1$.
Questions:

- What additional conditions on f from above will guarantee that $L = \lim_{n\to\infty} \frac{f(n)}{n}$ exists.

Conjectures, Open Problems, Further Investigations

Given: $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n) < n$ and $|f(n+1) - f(n)| \leq 1$.
Questions:

- What additional conditions on f from above will guarantee that $L = \lim_{n\to\infty} \frac{f(n)}{n}$ exists.
- What values of L are possible?

Conjectures, Open Problems, Further Investigations

Given: $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n) < n$ and $|f(n+1) - f(n)| \leq 1$.
Questions:

- What additional conditions on f from above will guarantee that $L = \lim_{n\to\infty} \frac{f(n)}{n}$ exists.
- What values of L are possible?
- Given such a function f, is there a sequence for which f is the growth function?

Conjectures, Open Problems, Further Investigations

Given: $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n) < n$ and $|f(n+1) - f(n)| \leq 1$.
Questions:

- What additional conditions on f from above will guarantee that $L = \lim_{n\to\infty} \frac{f(n)}{n}$ exists.
- What values of L are possible?
- Given such a function f, is there a sequence for which f is the growth function?
- Given a value of $0 \leq L \leq 1$, is there a growth function f for which $\lim_{n\to\infty} f(n)/n = L$?

QUESTIONS?

Contact information:

mike.brilleslyper@usafa.edu

brad.warner@usafa.edu