Power Fibonacci Sequences

Marc Renault

Shippensburg University

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About this work…

- This was work conducted with undergraduate student **Josh Ide**, who will start at SUNY Binghamton this fall for graduate studies

- "Power Fibonacci Sequences"
  Fibonacci Quarterly, May 2012

- Copies available

Fibonacci sequence, mod m

- My master's thesis, 1996
- F = 0, 1, 1, 2, 3, 5, 8, 13, 21, …
- $F \pmod 4$ = 0, 1, 1, 2, 3, 1, 0, 1, 1, …
- $F \pmod 5$ =

  0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, …

Fibonacci sequences mod m

- Mod 5:
  - 0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, …
  - 1, 3, 4, 2, 1, 3, 4, 2, …
  - 0, 0, …

$1, 3, 3^2, 3^3, 3^4, \ldots$

- Mod 6:
  - 0, 1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1, 0, 1…
  - 0, 2, 2, 4, 0, 4, 4, 2, 0, 2, 2, …
  - 0, 3, 3, 0, 3, 3, …
  - 0, 0, …
- Mod 11:
  - 0, 1, 1, 2, 3, 5, 8, 2, 10, 1, 0, 1, 1, …
  - ⋮
  - 1, 4, 5, 9, 3, 1, 4, …
  - 1, 8, 9, 6, 4, 10, 3, 2, 5, 7, 1, 8, 9, …
  - ⋮

$1, 4, 4^2, 4^3, 4^4, \ldots$

$1, 8, 8^2, 8^3, 8^4, \ldots$

Power Fibonacci Sequences

For which moduli do there exist power Fibonacci sequences?

- Must solve $x^2 \equiv x + 1 \pmod{m}$
- The golden ratio! $x \equiv \frac{1+\sqrt{5}}{2} \pmod{m}$
- Does $\sqrt{5}$ exist, modulo **a prime** p?
  - If $p = 5$, there is one solution, 0.
  - If $p \equiv \pm 1 \pmod{10}$, there are two distinct sol's
  - If $p \equiv \pm 3 \pmod{10}$, there are no solutions.
- For $p = 11$, $\sqrt{5} = 4 \text{ or } 7$, $\frac{1+\sqrt{5}}{2} = 8 \text{ or } 4$

Law of Quadratic Reciprocity

For which moduli do there exist power Fibonacci sequences?

- Does $\sqrt{5}$ exist modulo **a prime power** p^e?

- For $p = 5$:
  - Mod 25, $\sqrt{5}$ does not exist,
  - So, $\sqrt{5}$ does **not** exist mod 5^e for $e > 1$.

- For $p \equiv \pm 1 \pmod{10}$:
  - $\sqrt{5}$ has two distinct solutions mod p^e for $e \geq 1$.

Hensel's Lemma

For which moduli do there exist power Fibonacci sequences?

- Does $\sqrt{5}$ exist modulo **a composite** m?

- Suppose $\gcd(m_1, m_2) = 1$.
  If a congruence has n_1 solutions mod m_1,
  and n_2 solutions mod m_2,
  then it has $n_1 n_2$ solutions mod $m_1 m_2$

- Thus…

Chinese Remainder Theorem

For which moduli do there exist power Fibonacci sequences?

Theorem:

There is exactly one power Fibonacci sequence modulo 5.

For $m \neq 5$, there exist power Fibonacci sequences precisely when m has prime factorization

$$m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \quad \text{or} \quad m = 5p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

where each $p_i \equiv \pm 1 \pmod{10}$.

In this case, there are exactly 2^k power Fibonacci sequences.

Other Results

Period of power Fibonacci Sequences

For a given modulus m, let π denote the period of F. Either

- All PFS's have period π, or
- One PFS has period $\frac{1}{2}\pi$, all others have period π.

Subsequences

- In certain cases one PFS is a subsequence of the other.

Thank You!

Marc Renault

Shippensburg University

msrenault@ship.edu